Semi-random greedy independent set algorithm

He Guo* and Lutz Warnke†

June 5, 2025

Abstract

We study a semi-random (Rödl nibble) variant of the random greedy independent set algorithm to construct a large independent set in a hypergraph. Let $r \geq 2$ be a fixed integer and let $\mathcal{H}$ be a r-uniform N-vertex hypergraph satisfying that each vertex is contained in at most D edges. Assuming $\min\{N^{r-1}/D, D\} \geq f$ for some $f \geq (\log N)^c$ with $c > 0$, we prove that if $\mathcal{H}$ satisfies some degree and codegree conditions, then with high probability there are $\Omega\Big(N\big(\frac{\log f}{D}\big)^{\frac{1}{r-1}}\Big)$ vertices in the independent set constructed by the algorithm.

A key improvement of this semi-random variant compared to Bennett and Bohman's previous random greedy version is that we replace their requirement of $\mathcal{H}$ being D-regular by the maximum degree condition. We also prove that the independent set constructed by this algorithm has various pseudo-random properties, including those needed for applications in Ramsey theory. And we prove that the same holds for a random subhypergraph, i.e., the above properties remain true in the sparse setting.

Introduction

1 Introduction

A *hypergraph* $\mathcal{H}$ is a pair of finite sets (V, E), where $V_{\mathcal{H}} := V$ is called the *vertex set* of $\mathcal{H}$ and $E_{\mathcal{H}} := E$ is a collection of subsets of V and called the *edge set* of $\mathcal{H}$. An *independent set* of a hypergraph $\mathcal{H}$ is a vertex-subset of $V_{\mathcal{H}}$ that contains no edge of $\mathcal{H}$.

Numerous problems in combinatorics can be stated in terms of independent sets in hypergraphs. For example, let a hypergraph $\mathcal{H}$ have vertex-set $[n]$ and edge-set the collection of all k-term arithmetic progressions in $[n]$. Then the famous Szemerédi's theorem [23] says that the independent number of such hypergraph is $o(n)$. For more examples, see the recent breakthroughs using hypergraph containers [4, 20]. For this reason, lots of efforts have been put in to construct large independent set in hypergraphs (see, e.g., [1, 14, 6]).

In 2013, Bennett and Bohman [6] studied the behavior of a *random greedy independent set algorithm* applied to given (deterministic) hypergraph $\mathcal{H}$: the algorithm is defined by starting with an empty vertex-set and then iteratively adding one vertex of $\mathcal{H}$ at each step, chosen uniformly at random subject to the constraint that no edge of $\mathcal{H}$ is formed. They prove that if an r-uniform N-vertex hypergraph $\mathcal{H}$ is D-regular and satisfies some more conditions, then the algorithm with high probability provides an independent set of size $\Omega(N(\frac{\log N}{D})^{\frac{1}{r-1}})$. Let I_i be the independent set constructed at i step of the algorithm. They also proved that for a vertex set W of constant size containing no edge of $\mathcal{H}$, the distribution of $W \subseteq I_i$ obeys some pseudo-random properties for each i.

In this paper we extend their result: we relax regularity condition to maximum degree condition. Furthermore, we obtain additional pseudo-random properties, which allows us to prove bounds on some types of Ramsey numbers. For example, this way we obtain the following result (which seems hard to obtain from a direct analysis of H-free process).

randomKsfree

Corollary 1 (Sample application: pseudo-random K_s-free subgraphs)**.** *Given $s \geq 4$, there exist $\xi_0 = \xi_0(s)$ such that for all $\delta, \gamma \in (0, 1]$, $\xi \in (0, \xi_0]$ and $L \geq L_0(\delta, \gamma, \xi, H)$, the following holds for all $n \geq n_0(\delta, \gamma, \xi, L)$*

*Department of Mathematics and Mathematical Statistics, Umeå University, Umeå 90187, Sweden. E-mail: he.guo@umu.se.
Research supported by the Kempe Foundation grant JCSMK23-0055.

†Department of Mathematics, University of California, San Diego, La Jolla CA 92093, USA. E-mail: lwarnke@ucsd.edu.
Research supported by NSF grant DMS-1703516, NSF CAREER grant DMS-1945481, and a Sloan Research Fellowship.

with $\rho := \xi n^{-2/(s+1)}(\log n)^{1/(\binom{s}{2}-1)}$*. For any* n*-vertex graph* F*, there exists a* K_s*-free subgraph* $G \subseteq F$ *on the same vertex-set such the number of edges satisfies*

$$e_{G[W]} = (1 \pm \delta)\rho e_{F[W]}$$

for all vertex-sets $W \subseteq V_F$ *with* $|W| = \lceil L(\log n)^{1-1/(\binom{s}{2}-1)} n^{2/(s+1)} \rceil$ *and* $e_{F[W]} \geq \gamma\binom{|W|}{2}$*.*

ec:introISET

1.1 Main result: Independent sets in hypergraphs

To state our main result, we need to introduce some hypergraph parameters. For $r \geq 2$, let $\mathcal{H} = (V, E)$ be an r-uniform hypergraph with vertex-set V and edge-set $E_{\mathcal{H}} \subseteq \binom{V}{r}$. We define the maximum a-degree parameter

$$\Delta_a = \Delta_a(\mathcal{H}) := \max_{A \subseteq V_{\mathcal{H}} : |A| = a} \big|\{e \in E_{\mathcal{H}} : A \subseteq e\}\big|, \tag{1}$$

eq:def:adegr

which counts the maximum number of edges of $\mathcal{H}$ containing any a-element vertex-set of $\mathcal{H}$. Furthermore, we define the somewhat technical maximum $(r-1)$-codegree parameter

$$\Gamma(\mathcal{H}) := \max_{v, v' \in V_{\mathcal{H}} : v \neq v'} \big|\{(e, e') \in E_{\mathcal{H}} \times E_{\mathcal{H}} :\ v \in e \setminus e',\ v' \in e' \setminus e,\ |e \cap e'| = r-1\}\big|, \tag{2}$$

eq:def:codeg

which, for any two distinct vertices v, v' of $\mathcal{H}$, is an upper bound on the number of $(r-1)$-element "common neighbor" vertex-sets $S \subseteq V_{\mathcal{H}}$ such that $\{v\} \cup S$ and $\{v'\} \cup S$ are both edges of $\mathcal{H}$.

thm:main

Theorem 2 (Main Result)**.** *Fix* $r \geq 2$*. Let* $\mathcal{H} = (V, E)$ *be a* r*-uniform* N*-vertex hypergraph. For any* $\beta > 0$*, there exist constants* $\xi, c > 0$ *(depending on* β *and* r*) such that the following holds. If* $\mathcal{H}$ *satisfies*

$$\Delta_1(\mathcal{H}) \leq D, \tag{3}$$

$$\Delta_a(\mathcal{H}) \leq D^{\frac{r-a}{r-1}}/f \qquad \text{for } 1 < a \leq r-1, \tag{4}$$

$$\Gamma(\mathcal{H}) \leq D/f, \tag{5}$$

eq:assumptio
eq:assumptio
eq:assumptio

for suitable parameters D *and* f *that also satisfy*

$$f \geq (\log N)^c \qquad \text{and} \qquad \min\Big\{D^{\frac{1}{r-1}},\ N/D^{\frac{1}{r-1}}\Big\} \geq f^{\beta}, \tag{6}$$

eq:introassu

then the semi-random independent set algorithm produces an independent set $I \subseteq V$ *of* $\mathcal{H}$ *of size*

$$|I| \geq \xi N\Big(\frac{\log f}{D}\Big)^{\frac{1}{r-1}} \tag{7}$$

eq:thm:main

with probability at least $1 - N^{-\omega(1)}$*.*

Let V_p denote the binomial random vertex-subset of V, i.e., where each vertex of V is included independently with probability p.

randomsubset

Corollary 3 (Strengthening to random subsets of vertices)**.** *Fix* $r \geq 2$ *and* $\zeta > 0$*. Then, for any* $\beta > 0$*, there exist constants* $\xi, c > 0$ *(depending on* β*,* r *and* ζ*) such that for all* $p = p(n)$ *satisfying* $D^{-\frac{1}{r-1}} f^{\zeta} \leq p \leq 1$ *the following holds: after replacing the inclusion* $I \subseteq V$ *with* $I \subseteq V_p$*, the conclusion of Theorem 2 remains valid.*

The proof of this corollary is in Section 3.4.

Recall that a graph H is *strictly 2-balanced* if

$$\frac{e_H - 1}{v_H - 2} > \frac{e_{H'} - 1}{v_{H'} - 2}$$

for any $H' \subsetneq H$ with $v_{H'} \geq 3$. It is well-known that any graph $H \in \mathfrak{F}$ defined in (10) later is strictly 2-balanced. The above definition and the following lemma can be generalized to k-uniform hypergraphs H that are strictly k-balanced (see [9, Section 3]), which serves as an important class of hypergraphs for applications. We record this fact for later reference, and include the proof in Appendix A.1 for completeness.

tly2balanced

Lemma 4. *Let* H *be a strictly* 2*-balanced graph with minimum degree at least two. Then there exist constants* $\tau, A, n_0 > 0$ *(depending on* H*) such that, for any* $n \geq n_0$*, any subhypergraph* $\mathcal{H} \subseteq \mathcal{H}_H$ *with vertex-set* $V(\mathcal{H}) = V(\mathcal{H}_H)$ *satisfies the assumptions (3)–(6) of Theorem 2 with* $r = e_H$*,* $D := An^{v_H - 2}$ *and* $f := n^{\tau}$*.*

1.2 Pseudo-random properties and Applications

seudo-random

1.2.1 Pseudo-random vertex-distribution

As another application illustrating the pseudo-randomness of the constructed independent set I, the following result essentially states that each vertex is included into I with approximately same probability and there is some independence among them.

distribution

Theorem 5 (Pseudo-random distribution of the independent set)**.** *Fix $r \geq 2$ and $L \geq 1$. Let $\mathcal{H} = (V, E)$ be an r-uniform hypergraph satisfying the assumptions of Theorem 2. Then, for all $W \subseteq V$ of size $|W| \leq L$ that do not contain an edge of $\mathcal{H}$, the independent set I constructed by the semi-random independent set algorithm satisfies*

$$\mathbb{P}(W \subseteq I) \;=\; (1+o(1))\varrho^{|W|} \qquad \text{with} \qquad \varrho := \xi\left(\frac{\log f}{D}\right)^{\frac{1}{r-1}}, \tag{8}$$

eq:thm:verte

where $\xi, f, D > 0$ are as in Theorem 2.

Note that the estimate (8) intuitively says that we can think of I as a random subset of $V_{\mathcal{H}}$, where each vertex is included independently with probability ϱ. This estimate allows us to transfer Second Moment Methods proofs for such random subsets to the semi-random greedy independent set algorithm. To illustrate this, we consider a t-uniform hypergraph $\mathcal{G}$ on the vertex-set of $\mathcal{H}$, and let $X_{\mathcal{G}} := |E(\mathcal{G}[I])|$ denote the number of edges in $\mathcal{G}$ that are completely contained in the independent set I constructed by the semi-random independent set algorithm. The following corollary of Theorem 5 says that the number $X_{\mathcal{G}}$ of these edges is concentrated around its mean under fairly natural conditions (note that edges of $\mathcal{G}$ that are contained in $\mathcal{H}$ can impossibly be contained in the independent set I).

seudo-random

Corollary 6. *Fix $r, t \geq 2$. Let $\mathcal{H} = (V, E)$ be an r-uniform hypergraph satisfying the assumption of Theorem 2. Let $\mathcal{G}$ be a t-uniform hypergraph on the vertex-set V, such that no edge of $\mathcal{G}$ contains an edge of $\mathcal{H}$. If $|E(\mathcal{G})| \cdot \varrho^t \to \infty$ and $\Delta_a(\mathcal{G}) = o\big(|E(\mathcal{G})| \cdot \varrho^a\big)$ for $1 \leq a \leq t-1$, then with high probability (i.e., with probability tending to one as $N \to \infty$) we have*

$$X_{\mathcal{G}} \;=\; (1+o(1))|E(\mathcal{G})|\varrho^t, \tag{9}$$

eq:cor:pseud

where $\xi, f, D > 0$ are as in Theorem 2, and $\varrho = \varrho(\xi, f, D, r) > 0$ is defined as in (8).

The proof of this corollary is by a second-moment argument and is given in Section 5.

1.2.2 Construction of pseudo-random H-free graphs

Let $\mathfrak{F}$ be the collection of all graphs that are either a complete graph K_ℓ with $\ell \geq 4$, a cycle C_ℓ with $\ell \geq 4$, a complete bipartite graph $K_{a,a}$ with $a \geq 4$, or a hypercube Q^k with odd $k \geq 5$, i.e.,

$$\mathfrak{F} := \Big\{K_\ell : \ell \geq 4\Big\} \cup \Big\{C_\ell : \ell \geq 4\Big\} \cup \Big\{K_{a,a} : a \geq 4\Big\} \cup \Big\{Q^k : \text{odd } k \geq 5\Big\}. \tag{10}$$

def:Ffrak

As a general form of Corollary 1, we have the following theorem as an application of the pseudo-randomness in the constructed independent set.

orandomHfree

Theorem 7 (Pseudo-random H-free graphs)**.** *Given $H \in \mathfrak{F}$, there exist $\xi_0(H)$ such that for all $\delta, \gamma \in (0,1]$, $\xi \in (0, \xi_0]$ and $L \geq L_0(\delta, \gamma, \xi, H)$, the following holds for all $n \geq n_0(\delta, \gamma, \xi, L)$ with $\rho := \xi n^{-(v_H-2)/(e_H-1)}(\log n)^{1/(e_H-1)}$. For any n-vertex graph F, there exists an H-free subgraph $G \subseteq F$ on the same vertex-set such the number of edges satisfies*

$$e_{G[W]} = (1 \pm \delta)\rho e_{F[W]} \tag{11}$$

eq:thmpseudo

for all vertex-sets $W \subseteq V_F$ with $|W| = \lceil L n^{(v_H-2)/(e_H-1)}(\log n)^{1-1/(e_H-1)} \rceil$ and $e_{F[W]} \geq \gamma\binom{|W|}{2}$.

We provide the proof of Theorem 7 in Section 4.2.

We can restrict the H-free graph G to be in F_p.

cor:randomH

Corollary 8. *Given $H \in \mathfrak{F}$ and $\zeta > 0$, there exists $\xi_0 = \xi_0(H, \zeta) > 0$ such that for all $\delta, \gamma \in (0,1]$, $\xi \in (0, \xi_0]$ and $L \geq L_0(\delta, \gamma, \xi, H)$, the following holds for all $p = p(n)$ satisfying $n^{-(v_H-2)/(e_H-1)+\zeta} \leq p \leq 1$: after replacing the inclusion $G \subseteq F$ with $G \subseteq F_p$, the conclusion of Theorem 7 remains valid.*

We give the proof of Corollary 8 in Section 4.4.

1.2.3 Applications in Extremal Combinatorics

One can naturally extend the auxiliary hypergraph $\mathcal{H}_{H,F}$ discussed for graphs to k-uniform hypergraphs. It is well-known (see [6, Section 2]) that $\mathcal{H}_{H,K_n^{(k)}}$ satisfies the assumptions of Theorem 2 for a large family of k-uniform hypergraphs H, where $K_n^{(k)}$ denote the complete n-vertex k-uniform hypergraph with vertex-set $V(K_n^{(k)}) = [n]$ and edge-set $E(K_n^{(k)}) = \binom{[n]}{k}$. A k-unform hypergraph H is called strictly k-balanced if $(e_F - 1)/(v_F - k) < (e_H - 1)/(v_H - k)$ for all induced subhypergraphs $F \subsetneq H$ with $v_F > k$.

lem:HfreeHG **Lemma 9.** *Let H be a k-uniform hypergraph that has minimum degree at least two and is strictly k-balanced. Then there exist constants $\tau, A, n_0 > 0$ (depending on H and k) such that, for any $n \geq n_0$, any sub-hypergraph $\mathcal{H} \subseteq \mathcal{H}_H$ with vertex-set $V(\mathcal{H}) = V(\mathcal{H}_{H,K_n^{(k)}})$ satisfies the assumptions (3)–(6) of Theorem 2 with $D := An^{v_H - k}$ and $f := n^{\tau}$.*

The *graph Ramsey number* $R(H, G)$ is the minimum number n such that for any red and blue edge coloring of a complete graph K_n, there is either a red copy of H or a blue copy of G. Bohman–Keevash [8] prove that when H is a fixed complete graph or a cycle, then $R(H, K_t) = \Omega\Big(\frac{t^{(e_H-1)/(v_H-2)}}{(\log t)^{(e_H-2)/(v_H-2)}}\Big)$. We can extend this result as an immediate implication of Theorem 7 for $F = K_n$.

Corollary 10. *Given $H \in \mathfrak{F}$, we have*

$$R(H, K_t) = \Omega\left(\frac{t^{(e_H-1)/(v_H-2)}}{(\log t)^{(e_H-2)/(v_H-2)}}\right).$$

The *bipartite Ramsey number* $b(s, t)$ is the minimum number n such that for any red and blue edge coloring of a complete bipartite graph $K_{n,n}$, there is either a red copy of $K_{s,s}$ or a blue copy of $K_{t,t}$.

Caro–Rousseau [11] proved that for fixed s, we have $b(s, t) = \Omega\big((t/\log t)^{(s+1)/2}\big)$. Bal–Bennett [3] improved the logarithmic factor for $s = 2$ and obtained $b(2, t) = \Omega(t^{3/2}/\log t)$. Noting that $K_{2,2}$ is isomorphic to C_4, we extend the above results as an immediate conclusion of Theorem 7 for F to be the complete bipartite graph.

Corollary 11. *For fixed $s = 2$ or $s \geq 4$, we have*

$$b(s, t) = \Omega\Big((\log t)^{1/(2s-2)}(t/\log t)^{(s+1)/2}\Big).$$

We actually prove something more general with weaker assumption than the one on W in Theorem 7. See Section 4.2 for details.

1.3 Comparison with previous results

Regrading lower bound on the independence number of r-uniform hypergraphs, results were established by Ajtai, Komlós, Pintz, Spencer and Szemerédi [1], extended by Duke, Lefmann and Rödl [14], and later on by Bennett and Bohman [6]. In particular, Bennett and Bohman [6] get the following result.

nett2016note **Theorem 12** (Bennett–Bohman [6])**.** *Fix $r \geq 2$. Let $\mathcal{H} = (V, E)$ be a r-uniform, N-vertex hypergraph. For any $\epsilon > 0$, there exists a constant $\xi > 0$ (depending on ϵ and r) such that the following holds. If $\mathcal{H}$ is D-regular and satisfies assumptions (4)–(6) with $f = N^{\epsilon}$ and $\beta = 1$, then the random greedy independent set algorithm produces an independent set $I \subseteq V$ of $\mathcal{H}$ of size $|I| \geq \xi N\big((\log N)/D\big)^{\frac{1}{r-1}}$ with probability at least $1 - \exp\big\{-N^{\Omega(1)}\big\}$.*

Remark 13. *For $r = 2$ case, their proof (due to oversight) actually uses the assumption $N/D^{\frac{1}{r-1}} \geq N^{\beta}$ for some $\beta > 0$, which is omitted in the statement of the theorem in [6]. For $r \geq 3$, this condition can be obtained by the double-counting argument explained before. For the error probability, we can actually get $1 - \exp(-f^{\Omega(1)})$ in Theorem 2, which is same as Theorem 12 when $f = N^{\epsilon}$.*

They also prove with high probability, the output independent set I possesses some pseudo-random properties.

An improvement in our Theorem 2 compared to Theorem 12 is that we do not require $\mathcal{H}$ to be D-regular, but just have maximum degree at most D. Another strength is that we do not require the factor f in our theorem to be some power of N, but just some power of $\log N$.

Cooper and Mubayi [13] study the coloring problem in hypergraphs, and their result also implies a lower bound on the independence number similar as Theorem 2.

2016coloring **Theorem 14** (Cooper–Mubayi [13])**.** *Fix $r \geq 2$. Let $\mathcal{H} = (V, E)$ be a r-uniform N-vertex hypergraph. There exist a constant $\xi > 0$ (depending on r) such that the following holds. If $\mathcal{H}$ satisfies assumptions* (3) *and* (4) *for some parameters $D, f \geq 1$, then there exists an independent set $I \subseteq V$ of $\mathcal{H}$ of size $|I| \geq \xi N\big((\log f)/D\big)^{\frac{1}{r-1}}$.*

Remark 15. *They do not need the condition* (5) *and the bound on f.*

An improvement in our Theorem 2 compared to Theorem 14 is that their result only holds for uniformity $r \geq 3$ and just shows existence of the independent set without an efficient algorithm to construct it, while we can construct such an independent set with high probability for all $r \geq 2$. And another strength is that we study pseudo-random properties of the independent set, which is crucial for applications in Ramsey theory, while they did not.

2 Semi-random independent set algorithm

tionofnibble

In this section we formally define the semi-random (greedy) independent set algorithm that we use to find large independent sets in a given r-uniform hypergraph $\mathcal{H}$. As we shall see, the main result Theorem 2 of this paper follows from our main technical result Theorem 16 (see Section 2.5), which establishes various pseudo-random properties of the semi-random algorithm. This in turn requires us to introduce a number of auxiliary parameters, variables and definitions, which are spread across Sections 2.1–2.6.

We now start with a rough overview of the algorithm, since the formal details are somewhat technical. To construct an independent set in a hypergraph $\mathcal{H}$, the basic plan is to step-by-step build three kinds of vertex-subsets of $V_{\mathcal{H}}$: a 'random-like' set V_i (union of random chosen vertices up to step i), an independent set I_i of $\mathcal{H}$, and an 'available' set A_i (the vertices which may be added to V_{i+1} and I_{i+1} in the next step), satisfying

$$I_i \subseteq V_i, \tag{12}$$ eq:sizeofEiI

$$A_i \subseteq \Big\{ v \in V_{\mathcal{H}} \setminus V_i : \text{ there is no edge } e \in E_{\mathcal{H}} \text{ satisfying } e \setminus \{v\} \subseteq V_i \Big\}, \tag{13}$$ eq:requireme

where we will intuitively have $|I_i| \approx |V_i|$, with I_i having many features of a random vertex-subset of $V_{\mathcal{H}}$.

2.1 Auxiliary parameters

sec:aux

We start by introducing some auxiliary parameters. Set

$$\sigma := (\log f)^{-9r} \qquad \text{and} \qquad \rho := 1/2. \tag{14}$$ eq:def:sigma

Deferring the choice of the sufficiently small constant $\xi > 0$ to (37) in Section 2.6, with foresight we define the total number of steps of the algorithm as

$$m := \lfloor \xi (\log f)^{\frac{1}{r-1}} / \sigma \rfloor. \tag{15}$$ eq:defofm

For $0 \leq i \leq m$, we then define the parameters q_i and π_i as

$$\pi_i := \sigma + \sigma i, \tag{16}$$ eq:def:pii

$$q_i := e^{-(\sigma i)^{r-1}}, \tag{17}$$ eq:def:qi

and also define a 'relative error' parameter

$$\tau_i := 1 - (\log f)^{-1} \cdot \frac{\pi_i}{\pi_m}. \tag{18}$$

taui

Note that π_i is roughly σi (starting with $\pi_0 = \sigma > 0$ will be convenient later on). Furthermore, since $f \gg 1$, for $0 \le i \le m$ the relative error τ_i decays slowly from $1 - o((\log f)^{-1}\sigma)$ to $1 - (\log f)^{-1} = 1 - o(1)$.

2.2 Details of the semi-random algorithm

tdefofnibble

We now turn to the details of our nibble construction. In step $i = 0$, we set $\Gamma_0 := \varnothing$, and define

$$A_0 := V_{\mathcal{H}} \qquad \text{and} \qquad V_0 := I_0 := \varnothing. \tag{19}$$

def:V0E0

In step $i+1$ with $1 \le i+1 \le m = \lfloor \xi (\log f)^{\frac{1}{r-1}}/\sigma \rfloor$, we include each vertex $v \in A_i$ into Γ_{i+1} independently with probability[1]

$$p_i := \frac{\sigma}{q_i D^{\frac{1}{r-1}}}, \tag{20}$$

eq:def:pi

and then we set

$$V_{i+1} := V_i \cup \Gamma_{i+1}.$$

Note that $I_i \cup \Gamma_{i+1}$ may contain an edge of $\mathcal{H}$, recalling that some (one vertex is not enough by (13) above) vertices from Γ_{i+1} can form an edge with vertices in I_i, as formalized by

$$B_{i+1} := \Big\{ U \subseteq \Gamma_{i+1} : \text{ there exists } \hat{U} \subseteq I_i \text{ such that } U \cup \hat{U} \in E_{\mathcal{H}} \Big\}. \tag{21}$$

We ensure that I_{i+1} remains an independent set by alteration of $I_i \cup \Gamma_{i+1}$: denoting by $\mathcal{D}_{i+1}$ a maximal collection of vertex-disjoint elements of B_{i+1} (say the first one in lexicographic order to resolve ties), we set

$$I_{i+1} := I_i \cup \Big(\Gamma_{i+1} \setminus V(\mathcal{D}_{i+1}) \Big), \tag{22}$$

where we write $V(\mathcal{D}_{i+1}) := \bigcup_{\alpha \in \mathcal{D}_{i+1}} \alpha$. With an eye on constructing the available set $A_{i+1} \subseteq A_i \setminus \Gamma_{i+1}$, a close inspection of (13) reveals that we have to deal with some newly 'closed' vertices (i.e., vertices which form an edge with some other vertices in $V_{i+1} = V_i \cup \Gamma_{i+1}$). With this in mind, for any vertex $v \in A_0$ and integer $j = 1, \ldots, r-1$ we introduce

$$\begin{aligned} Y_{v,j}(i) := \big\{ W \subseteq A_0 \setminus \{v\} : |W| = j, \text{ and there exists } e \in \mathcal{H} \text{ such that} \\ v \in e, \ W \subseteq e \cap A_i, \text{ and } e \setminus (\{v\} \cup W) \subseteq V_i \big\}, \end{aligned} \tag{23}$$

def:setofYvj

where in this paper we shall (to avoid clutter) often use the abbreviation

$$Y_v(i) = Y_{v,1}(i).$$

For technical reasons we increase the set of closed vertices via the random set S_{i+1}, which is a random subset of A_i where each vertex $v \in A_i$ is included into S_{i+1} independently with probability

$$\hat{p}_{v,i} := 1 - (1-p_i)^{\max \big\{ (r-1) q_i \big(\pi_i^{r-2} + \sigma^{\rho} \pi_i^{\max\{r-3,0\}} \big) D^{\frac{1}{r-1}} - |Y_v(i)|, \, 0 \big\}}. \tag{24}$$

eq:hpvi

For $j = 1, \ldots, r-1$ we then introduce

$$C_{i+1}^{(j)} := \Big\{ v \in A_i : \text{ there exists } W \in Y_{v,j}(i) \text{ such that } W \subseteq \Gamma_{i+1} \Big\}, \tag{25}$$

eq:def:cCi+1

and define the 'closed' and available set of vertices as

$$\begin{aligned} C_{i+1} &:= C_{i+1}^{(1)} \cup S_{i+1}, \\ A_{i+1} &:= A_i \setminus \Big(\Gamma_{i+1} \cup C_{i+1} \cup \bigcup_{2 \le j \le r-1} C_{i+1}^{(j)} \Big), \end{aligned} \tag{26}$$

eq:closed:av

[1] Our setup and assumptions will guarantee that we indeed have $p_i \in (0,1)$, see Remark 19 in Section 2.6.

which ensures that A_{i+1} satisfies the inclusion property (13). The purpose of adding the 'self-stabilization' set S_{i+1} to C_{i+1} is that we intuitively want to make the 'closure' probability

$$\begin{aligned}\mathbb{P}(v \notin C_{i+1} \mid A_i, V_i) &= \mathbb{P}(v \notin C^{(1)}_{i+1} \mid A_i, V_i) \cdot (1-\hat{p}_{v,i}) \\ &= (1-p_i)^{\max\left\{(r-1)q_i\left(\pi_i^{r-2}+\sigma^{\rho}\pi_i^{\max\{r-3,0\}}\right)D^{\frac{1}{r-1}},\ |Y_v(i)|\right\}}\end{aligned} \tag{27}$$

eq:hope

equal for all available vertices $v \in A_i$ (and thus independent of history), by keeping track of the upper bound on $|Y_v(i)| = |Y_{v,1}(i)|$; see equation (50) and Lemma 27 in Section 3.1.1 for the details.

To summarize: at this point we have constructed the three vertex-subsets $V_{i+1}, I_{i+1}, A_{i+1}$ of $A_0 = V_{\mathcal{H}}$, and we iteratively repeat this construction until we obtain V_m, I_m, A_m at the end of the algorithm. In other words, the semi-random algorithm constructs the independent set $I := I_m$.

2.3 Pseudo-random heuristic for the size of I_m

domintuition

In this section we give a heuristic explanation for size $|I_m|$ of the resulting final independent set I_m, based on the heuristic that the vertex-sets A_i, V_i resemble some random subsets of A_0. In particular, our pseudo-random ansatz is that every vertex satisfies

$$\mathbb{P}(v \in V_i) \approx \frac{\pi_i}{D^{\frac{1}{r-1}}} \qquad \text{and} \qquad \mathbb{P}(v \in A_i) \approx q_i, \tag{28}$$

eq:qipii

where the different choices $v \in V_i$ are independent. This heuristic suggests $\pi_i \approx \sigma i$, since $\pi_0 \approx 0$ and

$$\frac{\pi_{i+1}-\pi_i}{D^{\frac{1}{r-1}}} \approx \mathbb{P}(v \in V_{i+1}) - \mathbb{P}(v \in V_i) \approx \mathbb{P}(v \in \Gamma_{i+1} \mid v \in A_i)\mathbb{P}(v \in A_i) \approx p_i q_i = \frac{\sigma}{D^{\frac{1}{r-1}}},$$

which is consistent with our definition (16) in Section 2.2. Recalling (13), the main requirement for $v \in A_i$ is that no edge e containing v satisfies $e \setminus \{v\} \subseteq V_i$. Since the hypergraph $\mathcal{H}$ has maximum degree at most D, our pseudo-random ansatz (combined with the Poisson paradigm) suggests that

$$q_i \approx \mathbb{P}(v \in A_i) \approx \mathbb{P}\Big(\bigcap_{e \in E_{\mathcal{H}}: v \in e} \{e \setminus \{v\} \not\subseteq V_i\}\Big) \geq \left(1 - \Big(\frac{\pi_i}{D^{\frac{1}{r-1}}}\Big)^{r-1}\right)^{D} \approx e^{-(\sigma i)^{r-1}}.$$

Using the self-stabilization mechanism discussed around (27), we will intuitively be able to show that this holds with $\geq$ replaced by $\approx$ (as one would expect if $\mathcal{H}$ was D-regular), which is consistent with our definition (17) from Section 2.2. Together with $|I_m| \approx |V_m|$ (see the heuristic description below (13)) as well as $\pi_m \approx \sigma m \approx \xi(\log f)^{\frac{1}{r-1}}$ and $|A_0| = |V_{\mathcal{H}}| = N$ (see (15) and (19)), this in turn makes it plausible that

$$|I_m| \approx |V_m| \approx \mathbb{E}(V_m) \approx \sum_{v \in A_0} \mathbb{P}(v \in V_m) \approx |A_0| \cdot \frac{\pi_m}{D^{\frac{1}{r-1}}} \approx \xi N \Big(\frac{\log f}{D}\Big)^{\frac{1}{r-1}},$$

which is consistent with the conclusion of Theorem 2 in Section 1 about the size of the resulting independent set $I = I_m$ (see Theorem 16 in Section 2.5 for a rigorous version of these heuristic considerations).

2.4 More auxiliary variables and parameters

b:defandpara

We now introduce some additional auxiliary variables and parameters. Recall that our goal is to track $|I_m|$. As we shall see, this requires us to track $|A_i|$ and $|V_i|$, which in turn requires us to bound the variables $|Y_{v,j}(i)|$ defined in (23), which in turn requires us to bound the following auxiliary variables:

$$\Delta_{j,\ell}(i) := \max_{J \subseteq A_0 : |J| = j} \Delta_{J,\ell}(i) \qquad \text{with} \qquad \Delta_{J,\ell}(i) := \sum_{e \in E_{\mathcal{H}} : J \subseteq e} \mathbb{1}_{\{|(e \setminus J) \cap V_i| \geq \ell\}}. \tag{29}$$

eq:def:cDjel

In our (upcoming) somewhat crude bounds on these auxiliary variables we shall use the parameters

$$P := D^{-\frac{1}{r-1}} Q \quad \text{with} \quad Q := f^{2\xi^{r-1}}, \tag{30}$$

def:P

where we intuitively think of P as an 'inclusion probability upper bound' (see (90)–(91) in Section 3.3) and $Q \gg (\log f)^{1/(r-1)}$ as an 'error term'.

andparameter

2.5 Main technical result: pseudo-random properties

In this section we state our main technical result Theorem 16 for the semi-random independent set algorithm, which implies our main result Theorem 2 and establishes various pseudo-random properties of the constructed vertex-subsets $(A_i, V_i, I_i, \Gamma_i, S_i)_{0 \le i \le m}$ of $A_0 := V_{\mathcal{H}}$.

Our main interest concerns the following event, which controls the size of the final independent set:

$$\mathcal{I}_m := \Big\{ |I_m| \in (1 \pm \delta)\xi |A_0| \Big(\frac{\log f}{D}\Big)^{\frac{1}{r-1}} \Big\}, \tag{31}$$

eq:def:event

where the small parameter $\delta > 0$ is to be specified in Theorem 16. As discussed in Section 2.4, in order to control $|I_m|$ we need to control a number of additional random variables. To take this into account, for each $0 \le i \le m$ we define the 'good' events

$$\mathfrak{X}_i := \mathcal{A}_i \cap \mathcal{Y}_i \cap \mathcal{N}_i \cap \mathcal{P}_i \qquad \text{and} \qquad \mathfrak{X}_{\le i} := \bigcap_{0 \le j \le i} \mathfrak{X}_j, \tag{32}$$

cGi

where the following events capture various pseudo-random properties of our algorithm:

$$\mathcal{A}_i := \Big\{ \tau_i q_i |A_0| \le |A_i| \le q_i |A_0| \Big\}, \tag{33}$$

eq:def:event

$$\mathcal{Y}_i := \Big\{ |Y_{v,j}(i)| \le \binom{r-1}{j} q_i^j \pi_i^{r-1-j} D^{\frac{j}{r-1}} \text{ for all } v \in A_0 \text{ and } 1 \le j \le r-1 \Big\}, \tag{34}$$

eq:def:event

$$\mathcal{N}_i := \Big\{ \Delta_{j,\ell}(i) \le D^{\frac{r-(j+\ell)}{r-1}} f^{-\mathbb{1}_{\{j+\ell<r\}}} Q^{\ell+1} \text{ for all } j, \ell \in \mathbb{N} \text{ with } 2 \le j \le r-\ell \Big\}, \tag{35}$$

Delta

$$\mathcal{P}_i := \Big\{ \sum_{J \in Y_{v,j}(i)} |Y_{v'}(i) \cap J| \le D^{\frac{j}{r-1}} f^{-1} Q^{2r-(j+1)} \text{ for all distinct } v, v' \in A_0 \text{ and } 1 \le j \le r-1 \Big\}. \tag{36}$$

Gamma

Note that the bounds in (31) and (33)–(36) are consistent with the pseudo-random intuition derived in Section 2.3; see (28) and also (30). For example, in (35) the variable $\Delta_{J,\ell}(i)$ is excepted to be less than $\Delta_j 2^r (Q/D^{\frac{1}{r-1}})^\ell$, since there are at most Δ_j edges f containing J, at most 2^r ways to choose more than ℓ vertices from f in V_i and each choice occurs with probability at most $(Q/D^{\frac{1}{r-1}})^\ell$. For simplicity we here discarded the constant 2^r by crudely enlarging the Q^ℓ factor to $Q^{\ell+1}$ in (35), which has negligible impact on our arguments since there is some elbow-room when we use this auxiliary bound on $\Delta_{J,\ell}(i)$.

We are now ready to state our main technical result regarding the semi-random independent set algorithm, which concretely says that the pseudo-random events $\mathcal{I}_m$ and $\mathfrak{X}_{\le m}$ both hold with very high probability. By considering the independent set $I := I_m$, in view of the event $\mathcal{I}_m$ from (31) and $|A_0| = |V_{\mathcal{H}}| = N$ we see that Theorem 16 with $\delta = 1/2$, say, implies our main result Theorem 2, after rescaling ξ in the lower bound (7) on $|I| = |I_m|$ by a factor of two, say. The simple proof of Corollary 3 is given in Section 3.4.

thm:boundofI

Theorem 16 (Main technical result)**.** *Fix $r \ge 2$. Let $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}})$ be a r-uniform N-vertex hypergraph satisfying the assumptions* (3)*–*(6) *of Theorem 2 for some $\beta \in (0, 1]$. Then there are $\xi, c > 0$ (depending on β and r) such that, whenever $\delta \ge (\log f)^{-1}$ in* (31)*, we have*

$$\mathbb{P}(\mathcal{I}_m \cap \mathfrak{X}_{\le m}) \ge 1 - N^{-\omega(1)}.$$

rem:step0

Remark 17. *The event $\mathfrak{X}_0 = \mathcal{A}_0 \cap \mathcal{Y}_0 \cap \mathcal{N}_0 \cap \mathcal{P}_0$ holds deterministically.*

sec:Choice

2.6 Technical estimates and choices of constants ξ, c

For concreteness, we now make the following choices of the constants ξ and c:

$$\xi := \Big(\frac{\beta}{37r}\Big)^{\frac{1}{r-1}} \qquad \text{and} \qquad c := \frac{2}{3\xi^{r-1}}. \tag{37}$$

eq:defofxian

For later reference (see Section 3.4 and 4) we remark that our arguments work with any choice of $\xi, c \in (0, \infty)$ that satisfy

$$\xi \le (\beta/(37r))^{\frac{1}{r-1}} \qquad \text{and} \qquad c \ge 2/(3\xi^{r-1}). \tag{38}$$

eq:cxismallc

We have made no effort to numerically optimize these constraints, which lead to the following convenient technical estimates (whose details can safely be ignored on a first reading: they are mainly for later reference). Recall that $\rho = 1/2$ by (14).

assumption67 **Lemma 18.** *If the assumptions of Theorem 16 hold, then $f \to \infty$ as $N \to \infty$, and the following estimates are valid for all $0 \le i \le m$:*

$$\max_{\substack{x \ge \rho, \\ 0 \le y \le 2r}} \sigma^x \pi_i^y = o\big((\log f)^{-2}\big), \tag{39}$$ eq:sigmapi

$$\frac{\max_{-6r \le j,k,x,y,z \le 6r} \sigma^x q_i^j \pi_i^y Q^k (\log f)^z}{\min\{f,\ f^\beta,\ D^{\frac{1}{r-1}}\}} \ll f^{-\beta/2}, \tag{40}$$ eq:assumptio

$$\min\{f^{\beta/2},\ Q\} \gg \log N. \tag{41}$$ eq:assumptio

Proof. Using (14)–(16), it is routine to establish estimate (39) by noting that

$$\max_{\substack{x \ge \rho, \\ 0 \le y \le 2r}} \sigma^x \pi_i^y \le \sigma^\rho \pi_m^{2r} \le (\log f)^{-9r/2} \cdot \big(2\xi (\log f)^{1/(r-1)}\big)^{2r} = o\big((\log f)^{-2}\big).$$

We next turn to estimate (40). Recalling (14)–(17) and (30), we have $1 \ge q_i \ge q_m \ge f^{-\xi^{r-1}} > 0$, $Q = f^{2\xi^{r-1}}$ and $\sigma^x, \pi_i^y = (\log f)^{O(1)}$, so that

$$f^{-\frac{\beta}{2}} \max_{-6r \le j,k,x,y,z \le 6r} \sigma^x q_i^j \pi_i^y Q^k (\log f)^z \le f^{-\frac{\beta}{2} + 6r \cdot \xi^{r-1} + 6r \cdot 2\xi^{r-1} + o(1)}.$$

The power of f in the last term above is strictly less than zero, since $36r\xi^{r-1} < \beta$ follows from (38). Combining this with the bounds $\min\{f, D^{\frac{1}{r-1}}\} \ge f^\beta$ and $0 < \beta \le 1$ from (6) establishes estimate (40).

Finally, note that (38) and $f \ge (\log N)^c$ also imply that $f^{\frac{\beta}{2}} \gg f^{2\xi^{r-1}} = Q \ge (\log N)^{4/3}$, which readily establishes estimate (41). □

Pprobability **Remark 19.** *Lemma 18 implies that p_i and P defined in (20) and (30) are well-defined probabilities, i.e., both satisfy $p_i, P \in (0,1)$.*

2.7 Concentration tools

:ToolsBounds

In this final preparatory section we gather (for later reference) some variants of the 'bounded differences' concentration inequality and standard Chernoff bounds, which we shall later use to prove various concentration bounds. On a first reading the reader may perhaps wish to skip straight to Section 3.

thm:BDI **Theorem 20** (Bounded differences inequality)**.** *Let $(\xi_\alpha)_{\alpha \in \mathcal{I}}$ be a finite family of independent random variables with $\xi_\alpha \in \{0,1\}$. Let $f : \{0,1\}^{|\mathcal{I}|} \to \mathbb{R}$ be a function, and assume that there exist numbers $(c_\alpha)_{\alpha \in \mathcal{I}}$ such that the following holds for all $z = (z_\alpha)_{\alpha \in \mathcal{I}} \in \{0,1\}^{|\mathcal{I}|}$ and $z' = (z'_\alpha)_{\alpha \in \mathcal{I}} \in \{0,1\}^{|\mathcal{I}|}$: $|f(z) - f(z')| \le c_{\alpha_0}$ if $z_\alpha = z'_\alpha$ for all $\alpha \ne \alpha_0$. Define $X := f\big((\xi_\alpha)_{\alpha \in \mathcal{I}}\big)$ and $\lambda := \sum_{\alpha \in \mathcal{I}} c_\alpha^2 \mathbb{P}(\xi_\alpha = 1)$. Then, for all $t \ge 0$,*

$$\mathbb{P}(X \ge \mathbb{E}X + t) \le \exp\left(-\frac{t^2}{2\lambda}\right) \tag{42}$$ eq:BDImon

if the function f is decreasing (i.e., that $f(z) \le f(z')$ whenever $z_\alpha \ge z'_\alpha$ for all $\alpha \in \mathcal{I}$).

rem:BDI:LT **Remark 21.** *Define $C := \max_{\alpha \in \mathcal{I}} c_\alpha$. If we drop the assumption that f is decreasing, then*

$$\max\Big\{\mathbb{P}(X \le \mathbb{E}X - t),\ \mathbb{P}(X \ge \mathbb{E}X + t)\Big\} \le \exp\left(-\frac{t^2}{2(\lambda + Ct)}\right). \tag{43}$$ eq:BDI

rem:chernoff **Remark 22** (Chernoff bounds)**.** *In the special case $X = \sum_{\alpha \in \mathcal{I}} \xi_\alpha$ we have $C = c_\alpha = 1$ and $\lambda = \mathbb{E}X$. Standard Chernoff bounds (or applying (42)–(43) to the decreasing function $-X$) then show that in this case $\mathbb{P}(X \le \mathbb{E}X - t)$ and $\mathbb{P}(X \ge \mathbb{E}X + t)$ are at most the right-hand side of (42) and (43), respectively.*

For random variables with a special combinatorial form involving occurrence of events with bounded 'overlaps', we need the following Chernoff-type upper tail estimate, which is a convenient corollary of a more general result by Warnke [26, Theorem 9]. Note that the exponent of (44) scales with $1/C$.

thm:UT **Theorem 23** (Limited dependencies upper tail inequality)**.** *Let $(\xi_a)_{a\in\mathcal{A}}$ be a finite family*[2] *of independent random variables with $\xi_a \in \{0,1\}$. Let $(Y_\alpha)_{\alpha\in\mathcal{I}}$ be a finite family of variables indexed by ordered pairs $\alpha = (\alpha_1, \alpha_2)$ with $Y_\alpha := \mathbb{1}_{\{\xi_a=1 \text{ for all } a\in\alpha_1\}}$ and $\mathbb{E}\sum_{\alpha\in\mathcal{I}} Y_\alpha \le \mu$. Define $Z_C := \max\sum_{\alpha\in\mathcal{J}} Y_\alpha$, where the maximum is taken over all $\mathcal{J}\subseteq\mathcal{I}$ with $\max_{\alpha\in\mathcal{J}} |\{\hat\alpha\in\mathcal{J} : \alpha_1\cap\hat\alpha_1 \neq \varnothing\}| \le C$. Then for all $C, t > 0$,*

$$\mathbb{P}(Z_C \ge \mu + t) \le \min\left\{\left(\frac{e\mu}{\mu+t}\right)^{(\mu+t)/C},\ \exp\left(-\frac{t^2}{2C(\mu+t)}\right)\right\}. \tag{44}$$ eq:C

rem:UT **Remark 24.** *Let $Y := \sum_{\alpha\in\mathcal{I}} Y_\alpha$. If $\mathcal{G}$ is an event that implies $|\{\hat\alpha\in\mathcal{I} : \alpha_1\cap\hat\alpha_1 \neq \varnothing,\ \xi_a = 1 \text{ for all } a\in\hat\alpha_1\}| \le C$ for all $\alpha\in\mathcal{I}$ with $Y_\alpha > 0$ (i.e. $\xi_a = 1$ for all $a\in\alpha_1$), then for μ and Z_C as defined in Theorem 23 we have*

$$\mathbb{P}(Y \ge \mu + t \text{ and } \mathcal{G}) \le \mathbb{P}(Z_C \ge \mu + t).$$

3 Analysis of the nibble

softhenibble

The goal of this section is to prove our main nibble result Theorem 16, which as discussed implies our main independence set result Theorem 2. To this end we shall prove the following auxiliary result Lemma 25, which consists of the three separate estimates (45)–(47) below (and a minor refinement of its proof also yields the stronger 'random vertex-subset' conclusion recorded in Corollary 3, see Section 3.4 for the details).

xiliarylemma **Lemma 25.** *Under the assumption of Theorem 16, with $\xi, c > 0$ satisfying (38), we have*

$$\max_{0\le i<m} \mathbb{P}\Big((\neg\mathcal{Y}_{i+1}\cup\neg\mathcal{A}_{i+1})\cap\mathcal{N}_{i+1} \;\Big|\; \mathfrak{X}_{\le i}\Big) \le N^{-\omega(1)}, \tag{45}$$ eq:eventofYv

$$\mathbb{P}\Big(\neg(\mathcal{N}_i\cap\mathcal{P}_i) \text{ for some } 0\le i\le m\Big) \le N^{-\omega(1)}, \tag{46}$$ eq:eventofCr

$$\mathbb{P}(\neg\mathcal{I}_m\cap\mathfrak{X}_{\le m}) \le N^{-\omega(1)}. \tag{47}$$ eq:eventofIt

Proof of Theorem 16 (assuming Lemma 25). Note that $m = (\log f)^{O(1)} \le N^{O(1)}$ by the assumed bound (6). Hence, since $\mathfrak{X}_0$ holds deterministically by Remark 17, in view of the good event $\mathfrak{X}_i = \mathcal{A}_i\cap\mathcal{Y}_i\cap\mathcal{N}_i\cap\mathcal{P}_i$ and inequalities (45)–(46) we readily obtain that

$$\begin{aligned}\mathbb{P}(\neg\mathfrak{X}_{\le m}) &\le N^{-\omega(1)} + \sum_{0\le i<m}\mathbb{P}\big(\neg\mathfrak{X}_{i+1}\cap\mathcal{N}_{i+1}\cap\mathcal{P}_{i+1}\cap\mathfrak{X}_{\le i}\big)\\ &\le N^{-\omega(1)} + \sum_{0\le i<m}\mathbb{P}\Big((\neg\mathcal{Y}_{i+1}\cup\neg\mathcal{A}_{i+1})\cap\mathcal{N}_{i+1}\;\Big|\;\mathfrak{X}_{\le i}\Big)\\ &\le O(m)\cdot N^{-\omega(1)} \le N^{-\omega(1)},\end{aligned} \tag{48}$$

which together with inequality (47) completes the proof of Theorem 16. □

:noneedextra **Remark 26.** *In the proof of Lemma 25, the constraint $N/D^{\frac{1}{r-1}} \ge f^\beta$ in (6) is only used in the proof of (47). Hence, for any hypergraph $\mathcal{H}$ satisfying the assumptions of Theorem 16 except for that constraint, for any choice of $\xi, c > 0$ satisfying (38), we have*

$$\mathbb{P}(\mathfrak{X}_{\le m}) \ge 1 - N^{-\omega(1)}.$$

The proof of Lemma 25 is spread across the remainder of this section, and its proof structure is as follows. Before proving the crude bounds (46) for $\mathcal{N}_i$ and $\mathcal{P}_i$ in Section 3.3 (see Theorem 35), in Section 3.1.1 we prove some preliminary lemmas regarding $\mathbb{P}(v\notin C_{i+1})$ as well as a generalization form of it (Lemma 27 and 29),

[2]In this paper, every time we apply Theorem 23, each index α has its unique first coordinate α_1, i.e., $\{\hat\alpha\in\mathcal{I} : \hat\alpha_1 = \alpha_1\} = \{\alpha\}$. Therefore in those cases we always identify α with α_1, i.e., we use singletons directly rather than pairs as indices in $\mathcal{I}$.

which reveal the technical benefit of the self-stabilization mechanism introduced around (24)–(27). At the beginning of Section 3.1.2 we bound $\sum_{u \in A_i} |Y_u(i) \cap J|$ from above (Lemma 30), which turns out to be closely related to the λ term (57) used in later applications of the bounded differences inequality (see Theorem 20 and Remark 21). In the next part of Section 3.1.2 we then bound $|Y_{v,j}(i+1)|$ from above (Theorem 31). This one-sided bound is enough to show concentration bounds for $|A_{i+1}|$ for $0 \le i < m$ in the second part of Section 3.1.2 (Theorem 32) and for the size of final independent set $|I_m|$ in Section 3.2 (Theorem 33), which together establish the remaining estimates (45) and (47) of Lemma 25.

3.1 One-step analysis: events $\mathcal{Y}_{i+1}$ and $\mathcal{A}_{i+1}$

sec:one-step

In this subsection, we prove estimate (45) of Lemma 25, which concerns the auxiliary events $\mathcal{A}_i$ and $\mathcal{Y}_i$ defined in (33) and (34). To avoid clutter, we often omit the conditioning from our notations: we often write $\mathbb{P}(\cdot)$ and $\mathbb{E}(\cdot)$ as an abbreviation of $\mathbb{P}(\cdot \mid \mathcal{F}_i)$ and $\mathbb{E}(\cdot \mid \mathcal{F}_i)$, where $(\mathcal{F}_i)_{0\le i\le m}$ is the natural filtration associated with $(A_i, V_i, I_i, \Gamma_i, S_i)_{0\le i\le m}$, as usual. And we also assume that the $\mathcal{F}_i$-measurable event $\mathfrak{X}_{\le i}$ holds. Note that conditional on $\mathcal{F}_i$, by the construction of the random vertex-sets Γ_{i+1} and S_{i+1}, the (conditional) probability space formally consists of $2|A_i|$ independent Bernoulli random variables $(\mathbb{1}_{\{u\in\Gamma_{i+1}\}}, \mathbb{1}_{\{u\in S_{i+1}\}})_{u\in A_i}$, with $\mathbb{P}(u \in \Gamma_{i+1}) = p_i = \sigma/(q_i D^{1/(r-1)})$ and $\mathbb{P}(u \in S_{i+1}) = \hat{p}_{u,i}$ (see (20) and (24) in Section 2.2).

3.1.1 Effect of self-stabilization: nearly equal 'closure' probabilities

tabilization

In intuitive words, the following key auxiliary estimate (49) says that the 'closure' probabilities $\mathbb{P}(v \notin C_{i+1})$ are approximately the same for all vertices $v \in A_i$. Here the self-stabilization mechanism introduced around (24)–(27) is crucial (see also (50) in the below proof): without this extra twist vertices with higher degree would tend to be closed with higher probability (as they are more likely to be contained in edges without this extra twist; cf. the heuristic discussion in Section 2.3). As we shall see, Lemma 27 and the variant Lemma 29 below will often be used together with $A_{i+1} \subseteq A_i \setminus C_{i+1}$ to estimate various expectations.

probnotinC **Lemma 27.** *For any vertex $v \in A_i$ we have*

$$\mathbb{P}(v \notin C_{i+1}) - \frac{q_{i+1}}{q_i} \in \Big[-\frac{3}{2}(r-1)\sigma^{1+\rho}\pi_i^{\max\{r-3,0\}},\ -\frac{1}{2}(r-1)\sigma^{1+\rho}\pi_i^{\max\{r-3,0\}}\Big]. \tag{49}$$

eq:estimatep

oundsofsigma **Remark 28.** *The proof also establishes $0 \le q_i - q_{i+1} \le q_i \cdot O(\sigma\pi_i^{r-2}) \le \sigma^{1/2} q_i$ by (53).*

Proof. Since $\mathfrak{X}_{\le i} \subseteq \mathcal{Y}_i$ implies $|Y_v(i)| \le (r-1)q_i\pi_i^{r-2}D^{\frac{1}{r-1}}$, by definition of $\hat{p}_{v,i}$ in (24) we have,

$$\mathbb{P}(v \notin C_{i+1}) = (1-p_i)^{|Y_v(i)|}(1-\hat{p}_{v,i}) = (1-p_i)^{(r-1)q_i\left(\pi_i^{r-2}+\sigma^{\rho}\pi_i^{\max\{r-3,0\}}\right)D^{\frac{1}{r-1}}}. \tag{50}$$

eq:probnotin

As we shall see, noting $p_i q_i D^{\frac{1}{r-1}} = \sigma$, this identity eventually implies inequality (49) by elementary (but slightly tedious) calculations; the reader may wish to skip the below details on a first reading.

Turning to the details, using that $|(1-x)^\kappa - (1-\kappa x)| \le \binom{\kappa}{2}x^2$ for all $\kappa \in [2,\infty)$ and $x \in [0,1]$, with (40) we infer

$$\begin{aligned}&\Big|\mathbb{P}(v \notin C_{i+1}) - \Big(1-(r-1)\sigma\pi_i^{r-2} - (r-1)\sigma^{1+\rho}\pi_i^{\max\{r-3,0\}}\Big)\Big| \\ &\qquad = O\Big(\sigma^2\pi_i^{2(r-2)} + \sigma^{2+\rho}\pi_i^{r-2+\max\{r-3,0\}} + \sigma^{2+2\rho}\pi_i^{2\max\{r-3,0\}}\Big).\end{aligned} \tag{51}$$

eq:probnotin

We now claim that

$$\Big|1-(r-1)\sigma\pi_i^{r-2} - \frac{q_{i+1}}{q_i}\Big| \le \sigma^2 \cdot O(\pi_i^{2(r-2)} + \mathbb{1}_{\{r\ge 3\}}\pi_i^{r-3}), \tag{52}$$

eq:probnotin

which together with (51) establishes inequality (49) by a short calculation (using $\pi_i = O(\log f)$, $\sigma = (\log f)^{-9r}$ and $\rho = 1/2$). Turning to the proof of (52), using Taylor's Theorem and $q(t) := e^{-t^{r-1}}$, in view of $q_i = q(i\sigma)$ and $\pi_i = (i+1)\sigma$ it is routine to check that

$$\Big|q_i - q_{i+1} + \sigma q'(i\sigma)\Big| \le \frac{\sigma^2}{2}\max_{\eta: i\sigma\le\eta\le(i+1)\sigma}|q''(\eta)| \le \sigma^2 \cdot O\Big(\pi_i^{2(r-2)} + \mathbb{1}_{\{r\ge3\}}\pi_i^{r-3}\Big)\cdot q_i. \tag{53}$$

eq:probnotin

Furthermore, by the Binomial Theorem and $\pi_i \geq \sigma$ we have (using the standard convention $0^0 = 1$)

$$\left|\sigma \frac{q'(i\sigma)}{q_i} - \Big(-(r-1)\sigma\pi_i^{r-2}\Big)\right| = (r-1)\sigma \cdot |(\pi_i - \sigma)^{r-2} - \pi_i^{r-2}| = \mathbb{1}_{\{r\geq 3\}} O(\sigma^2 \pi_i^{r-3}). \tag{54}$$

eq:probnotin

Dividing (53) by q_i, now inequality (52) follows readily, completing the proof (as discussed). □

tiprobnotinC

Lemma 29. *For all $1 \leq j \leq r-1$ and distinct vertices $w_1, \ldots, w_j \in A_i$, we have*

$$\mathbb{P}(w_1, \ldots, w_j \notin C_{i+1}) \leq \Big(\frac{q_{i+1}}{q_i}\Big)^j \Big(1 - \frac{1}{8}\sigma^{1+\rho}\pi_i^{\max\{r-3,0\}}\Big). \tag{55}$$

eq:probnotin

Proof. The case $j = 1$ follows from Lemma 27 and the fact $q_{i+1}/q_i \sim 1$ (see Remark 28).

We henceforth assume $j \geq 2$, which enforces $r \geq 3$. Similar to (50), using inclusion-exclusion, we have

$$\begin{aligned}\mathbb{P}(w_1, \ldots, w_j \notin C_{i+1}) =& (1-p_i)^{|Y_{w_1}(i) \cup \cdots \cup Y_{w_j}(i)|} \cdot \prod_{\ell=1}^{j} (1 - \hat{p}_{w_\ell, i}) \\ \leq & (1-p_i)^{-\sum_{1\leq s<t\leq j} |Y_{w_s}(i) \cap Y_{w_t}(i)|} \cdot \prod_{\ell=1}^{j} \mathbb{P}(w_\ell \notin C_{i+1}).\end{aligned}$$

Note that $\mathfrak{X}_{\leq i} \subseteq \mathcal{P}_i$ implies $|Y_{w_s}(i) \cap Y_{w_t}(i)| \leq \sum_{J \in Y_{w_s,1}(i)} |Y_{w_t}(i) \cap J| \leq D^{\frac{1}{r-1}} f^{-1} Q^{2r-2} =: y$ for $s \neq t$, and that $p_i \cdot y \ll \sigma^{1+\rho}\pi_i^{\max\{r-3,0\}} = o(1)$ by (39)–(40) in Lemma 18. It follows that

$$\mathbb{P}(w_1, \ldots, w_j \notin C_{i+1}) \leq \Big(1 + o\Big(\sigma^{1+\rho}\pi_i^{\max\{r-3,0\}}\Big)\Big) \prod_{\ell=1}^{j} \mathbb{P}(w_\ell \notin C_{i+1}),$$

which in view of Lemma 27 and $q_{i+1}/q_i \sim 1$ (see Remark 28) routinely establishes inequality (55). □

ablevertices

3.1.2 Events $\mathcal{Y}_{i+1}$ and $\mathcal{A}_{i+1}$: partial neighbors and available vertices

Conditional on $\mathcal{F}_i$, the random variables that we shall consider later on will be of form

$$X = f\Big(\big(\mathbb{1}_{\{v\in\Gamma_{i+1}\}}, \mathbb{1}_{\{v\in S_{i+1}\}}\big)_{v\in A_i}\Big), \tag{56}$$

eq:Xf

where $(\mathbb{1}_{\{v\in\Gamma_{i+1}\}}, \mathbb{1}_{\{v\in S_{i+1}\}})_{v\in A_i}$ are Bernoulli random variables with $\mathbb{P}(v \in \Gamma_{i+1}) = p_i = \sigma/(q_i D^{1/(r-1)})$ and $\mathbb{P}(v \in S_{i+1}) = \hat{p}_{v,i} \leq 1$ (see the discussion at the beginning of Section 3). In many cases we will use the bounded differences concentration inequality (see Theorem 20 and Remark 21) to bound the tails of the random variable X of form (56). For this purpose, we need to consider the following two parameters: we define the *vertex-effect* c_v as an upper bound on how much X can change if we modify the indicator $\mathbb{1}_{\{v\in\Gamma_{i+1}\}}$ (alter whether v is in Γ_{i+1} or not), and the *stabilization-effect* $\hat{c}_v$ as an upper bound on how much X can change if we modify the indicator $\mathbb{1}_{\{v\in S_{i+1}\}}$ (alter whether v is in S_{i+1} or not). For convenience, we will frequently estimate the associated parameter λ from Theorem 20 via

$$\lambda = \sum_{v\in A_i} c_v^2 p_i + \sum_{v\in A_i} \hat{c}_v^2 \hat{p}_{v,i} \leq p_i \sum_{v\in A_i} c_v^2 + \sum_{v\in A_i} \hat{c}_v^2, \tag{57}$$

eq:lambdainB

which can often be further bounded by bringing the following double-counting based estimate into play.

intersection

Lemma 30. *If $\mathfrak{X}_{\leq i}$ holds, then $\sum_{u\in A_i} |Y_u(i) \cap J| \leq (r-1)q_i\pi_i^{r-2}D^{\frac{1}{r-1}} \cdot |J|$ for any $J \subseteq A_0$.*

Proof. For $u \in A_i$, note that $w \in Y_u(i)$ implies $u \in Y_w(i)$. Hence

$$\sum_{u\in A_i} |Y_u(i) \cap J| = \sum_{w\in J}\sum_{u\in A_i} \mathbb{1}_{\{w\in Y_u(i)\}} \leq \sum_{w\in J}\sum_{u\in A_i} \mathbb{1}_{\{u\in Y_w(i)\}} \leq \sum_{w\in J} |Y_w(i)|,$$

which completes the proof, since $|Y_w(i)| \leq (r-1)q_i\pi_i^{r-2}D^{\frac{1}{r-1}}$ by $\mathfrak{X}_{\leq i} \subseteq \mathcal{Y}_i$. □

After these preparations we are now ready to prove estimate (45), and we start by bounding the number $|Y_{v,j}(i+1)|$ of partial neighbors from above, cf. (34). Here the basic proof idea is to bound $|Y_{v,j}(i+1)|$ via several more tractable auxiliary random variables (see (58) below), which in turn can be bounded by the concentration inequalities from Section 2.7 (with some technical effort) .

thm:Yvj **Theorem 31.** *We have* $\mathbb{P}(\neg\mathcal{Y}_{i+1} \cap \mathcal{N}_{i+1} \mid \mathcal{F}_i) \le N^{-\omega(1)}$ *when* $\mathfrak{X}_{\le i}$ *holds.*

Proof. As discussed, we shall henceforth omit the conditioning on $\mathcal{F}_i$ from our notation. Fix $1 \le j \le r-1$. In intuitive words, a set W of size j is in $Y_{v,j}(i+1)$ only if it either (a) 'stays' in $Y_{v,j}$, or (b) 'newly enters' $Y_{v,j}$. More formally, $W \in Y_{v,j}(i+1)$ implies that either (a) $W \in Y_{v,j}(i)$ and $W \cap C_{i+1} = \varnothing$, or (b) there is some $U' \in Y_{v,k}(i)$ with $U' \supseteq W$ and $U' \setminus W \subseteq \Gamma_{i+1}$. It follows that

$$|Y_{v,j}(i+1)| \le \underbrace{\sum_{W \in Y_{v,j}(i)} \mathbb{1}_{\{W \cap C_{i+1} = \varnothing\}}}_{=:Y^*_{v,j}} + \sum_{j+1 \le k \le r-1} \underbrace{\sum_{(U,U'):U' \in Y_{v,k}(i), U \subseteq U', |U| = k-j} \mathbb{1}_{\{U \subseteq \Gamma_{i+1}\}}}_{=:Y^+_{v,j,k}}. \qquad (58)$$ Yvji+1

Note that the right-hand side of (58) reduces to $Y^*_{v,j}$ for $j = r-1$. Since $\mathfrak{X}_{\le i} \subseteq \mathcal{Y}_i$ implies $|Y_{v,j}(i)| \le \binom{r-1}{j} q_i^j \pi_i^{r-1-j} D^{\frac{j}{r-1}}$, invoking Lemma 29, we obtain

$$\mathbb{E}Y^*_{v,j} \le \binom{r-1}{j} q_i^j \pi_i^{r-1-j} D^{\frac{j}{r-1}} \cdot \Big(\frac{q_{i+1}}{q_i}\Big)^j \Big(1 - \frac{1}{8}\sigma^{1+\rho}\pi_i^{\max\{r-3,0\}}\Big).$$

Furthermore, since $\mathfrak{X}_{\le i} \subseteq \mathcal{Y}_i$ implies $|Y_{v,k}(i)| \le \binom{r-1}{k} q_i^k \pi_i^{r-1-k} D^{\frac{k}{r-1}}$, it routinely follows (substituting $s = k-j$ in the last equality) that

$$\begin{aligned}\sum_{k=j+1}^{r-1} \mathbb{E}Y^+_{v,j,k} &\le \sum_{j+1 \le k \le r-1} |Y_{v,k}(i)| \binom{k}{k-j} p_i^{k-j} \\ &\le \sum_{j+1 \le k \le r-1} \binom{r-1}{k} q_i^k \pi_i^{r-1-k} D^{\frac{k}{r-1}} \cdot \binom{k}{k-j} p_i^{k-j} \qquad (59) \\ &= \binom{r-1}{j} q_i^j D^{\frac{j}{r-1}} \sum_{1 \le s \le r-1-j} \binom{r-1-j}{s} \sigma^s \pi_i^{r-1-j-s}.\end{aligned}$$ eq:EYvjk

Expanding $\pi_i^{r-1-j} = \pi_{i+1}^{r-1-j} - \Big((\pi_i + \sigma)^{r-1-j} - \pi_i^{r-1-j}\Big)$ via the Binomial Theorem, it follows that

$$\begin{aligned}\mathbb{E}Y^*_{v,j} + \sum_{j+1 \le k \le r-1} \mathbb{E}Y^+_{v,j,k} \le &\binom{r-1}{j} q_{i+1}^j \pi_{i+1}^{r-1-j} D^{\frac{j}{r-1}} - \frac{1}{8}\binom{r-1}{j} \sigma^{1+\rho} q_{i+1}^j \pi_i^{r-1-j+\max\{r-3,0\}} D^{\frac{j}{r-1}} \\ &+ \binom{r-1}{j} (q_i^j - q_{i+1}^j) D^{\frac{j}{r-1}} \sum_{1 \le s \le r-1-j} \binom{r-1-j}{s} \sigma^s \pi_i^{r-1-j-s}. \qquad (60)\end{aligned}$$ eq:EYvj

With the intention of simplifying these expressions, our next aim is to show that the last term of (60) is negligible compared to the second last one. Using that $q_i \sim q_{i+1} \ge q_i\Big(1 - O(\sigma\pi_i^{r-2})\Big)$ by Remark 28, and that $\sigma\pi_i^{r-2} \ll 1$ by (39), it follows that

$$q_i^j - q_{i+1}^j = q_i^j\left(1 - \Big(\frac{q_{i+1}}{q_i}\Big)^j\right) \le q_{i+1}^j O(\sigma\pi_i^{r-2}). \qquad (61)$$ eq:qij-qi+1j

With an eye on the sum (60) and the above estimate (61), note that $\pi_i \ge \sigma$ implies $\sigma\pi_i^{r-2} \cdot \sigma^s \pi_i^{r-1-j-s} \le \sigma^2 \pi_i^{r-1-j+r-3}$ for all $1 \le s \le r-1-j$ (which in turn also enforces $r \ge j+2 \ge 3$). Putting things together, using $\rho < 1$ it follows that

$$\binom{r-1}{j}(q_i^j - q_{i+1}^j) D^{\frac{j}{r-1}} \sum_{s=1}^{r-1-j} \binom{r-1-j}{s} \sigma^s \pi_i^{r-1-j-s} \ll \sigma^{1+\rho} q_{i+1}^j \pi_i^{r-1-j+\max\{r-3,0\}} D^{\frac{j}{r-1}}.$$

In concrete words, this establishes that on the right-hand side of (60), the last term is indeed negligible compared to the second last one (as intended). Defining with foresight

$$t := \frac{1}{16r}\binom{r-1}{j}\sigma^{1+\rho}q_{i+1}^{j}\pi_i^{r-1-j+\max\{r-3,0\}}D^{\frac{j}{r-1}}, \tag{62}$$

def:t

in view of estimates (58)–(60), the above discussion and (34), for the proof of Theorem 31 it thus remains to show that

$$\max_{v\in A_0}\Big[\mathbb{P}\Big(Y^*_{v,j}\geq \mathbb{E}Y^*_{v,j}+t\Big)+\sum_{j+1\leq k\leq r-1}\mathbb{P}\Big(Y^+_{v,j,k}\geq \mathbb{E}Y^+_{v,j,k}+t \text{ and } \mathcal{N}_{i+1}\Big)\Big]\leq N^{-\omega(1)}. \tag{63}$$

type12

For the upper bound on $Y^*_{v,j}$ in (63) we shall use the bounded differences inequality (Theorem 20), which requires us the to bound the parameter λ from (57) associated with $Y^*_{v,j}$, i.e.,

$$\lambda \leq p_i\sum_{u\in A_i}c_u^2+\sum_{u\in A_i}\hat{c}_u^2. \tag{64}$$

eq:lambdainB

The vertex-effect c_u (an upper bound on how much $Y^*_{v,j}$ changes if we alter whether $u\in\Gamma_{i+1}$ or $u\notin\Gamma_{i+1}$) satisfies $c_u\leq\sum_{J\in Y_{v,j}(i)}|Y_u(i)\cap J|$. Using $\mathfrak{X}_{\leq i}\subseteq\mathcal{P}_i$, by Lemma 30 and the estimate from (34) for $|Y_{v,j}(i)|$ it thus follows that

$$\sum_{u\in A_i}c_u\leq\sum_{u\in A_i}\sum_{J\in Y_{v,j}(i)}|Y_u(i)\cap J|=\sum_{J\in Y_{v,j}(i)}\sum_{u\in A_i}|Y_u(i)\cap J|\leq O(q_i^j\pi_i^{r-1-j}D^{\frac{j}{r-1}}\cdot q_i\pi_i^{r-2}D^{\frac{1}{r-1}}).$$

Furthermore, if $u\neq v$, then using $\mathfrak{X}_{\leq i}\subseteq\mathcal{P}_i$, we may apply the estimate from (36) to obtain $c_u\leq D^{\frac{-j}{r-1}}f^{-1}Q^{2r-(j+1)}$. In the case $u=v$, we claim that $c_v=O(q_i\pi_i^{r-2}D^{\frac{j}{r-1}})$ holds, which in case of $j=1$ is immediate from $\mathfrak{X}_{\leq i}\subseteq\mathcal{Y}_i$ and (34), since then

$$c_v\leq\sum_{J\in Y_{v,1}(i)}|Y_v(i)\cap J|=\sum_{w_1\in Y_v(i)}\sum_{w_2\in Y_v(i)}\mathbb{1}_{\{w_1=w_2\}}=|Y_v(i)|=O(q_i\pi_i^{r-2}D^{\frac{1}{r-1}}).$$

In the case $2\leq j\leq r-1$, using $\mathfrak{X}_{\leq i}\subseteq\mathcal{Y}_i\cap\mathcal{N}_i$ and (34)–(35), we obtain that

$$c_v\leq\sum_{J\in Y_{v,j}(i)}\sum_{w_1\in J}\sum_{w_2\in Y_v(i)}\mathbb{1}_{\{w_1=w_2\}}\leq|Y_v(i)|\cdot\Delta_{2,r-j-1}(i)=O(q_i\pi_i^{r-2}D^{\frac{j}{r-1}}f^{-1}Q^{r-j}),$$

since for any $w\in Y_v(i)$ with $w\in J$ for some $J\in Y_{v,j}(i)$, there is an edge $e\in E_{\mathcal{H}}$ containing $\{v,w\}$ with $|(e\setminus\{w,v\})\cap V_i|\geq r-j-1$ (as $v\neq w$ must hold when $w\in J\in Y_{v,j}(i)$ by definition (23)). The number of such edges is at most $\Delta_{2,r-j-1}(i)$ (see (29) for definition of this parameter), and each such edge can contain at most one element from $Y_{v,j}(i)$. Noting that (40) and $\beta\leq 1$ imply $f^{-1}Q^{r-j}\ll 1$ then completes the proof of the claimed bound $c_v=O(q_i\pi_i^{r-2}D^{\frac{j}{r-1}})$. Putting the above-discussed estimates together, it follows that

$$\begin{aligned}\sum_{u\in A_i}c_u^2&\leq\Big(\max_{u\in A_i:u\neq v}c_u\Big)\cdot\sum_{u\in A_i}c_u+c_v^2\\&\leq O\Big(D^{\frac{-j}{r-1}}f^{-1}Q^{2r-(j+1)}\cdot q_i^{j+1}\pi_i^{2r-3-j}D^{\frac{j+1}{r-1}}\Big)+O\Big(\big(q_i\pi_i^{r-2}D^{\frac{j}{r-1}}\big)^2\Big).\end{aligned}$$

Recall the definition (62) of t, and that $p_i=\sigma/(q_iD^{\frac{1}{r-1}})$ by (20). Comparing the power of D (which is the main term), by noting the presence of f^{-1} and using inequality (40) from Lemma 18 together with the estimate $q_i\sim q_{i+1}$ from Remark 28, it follows that

$$p_i\sum_{u\in A_i}c_u^2=O\big(f^{-\beta/2}t^2\big).$$

Note that the stabilization-effect $\hat{c}_w$ (an upper bound on how much $Y^*_{v,j}$ changes if we alter whether $w\in S_{i+1}$ or $w\notin S_{i+1}$) satisfies $\hat{c}_w\leq\sum_{J\in Y_{v,j}(i)}\mathbb{1}_{\{w\in J\}}\leq\Delta_{2,r-j-1}(i)$, since for any $J\in Y_{v,j}(i)$ with $w\in J$, there is an edge $e \in \mathcal{H}$ containing $\{w, v\}$ with $|(e \setminus \{w, v\}) \cap V_i| \geq r - j - 1$ (as $w \neq v$ by definition of $Y_{v,j}(i)$ in (23)). The number of such edges is at most $\Delta_{2,r-j-1}(i)$ (see (29) for definition of this parameter), and each such edge can contain at most one element from $Y_{v,j}(i)$. Using $\mathfrak{X}_{\leq i} \subseteq \mathcal{Y}_i \cap \mathcal{N}_i$ to bound $|Y_{v,j}(i)|$ and $\Delta_{2,r-j-1}(i)$ via (34)–(35), together with inequality (40) from Lemma 18 and $q_i \sim q_{i+1}$ from Remark 28 it follows that

$$\begin{aligned}\sum_{u \in A_i} \hat{c}_u^2 &\leq \Big(\max_{u \in A_i} \hat{c}_u\Big) \cdot \sum_{u \in A_i} \sum_{J \in Y_{v,j}(i)} \mathbb{1}_{\{u \in J\}} \\ &\leq \Delta_{2,r-j-1}(i) \cdot j \cdot |Y_{v,j}(i)| \\ &\leq D^{\frac{j-1}{r-1}} Q^{r-j} \cdot j \cdot \binom{r-1}{j} q_i^j \pi_i^{r-1-j} D^{\frac{j}{r-1}} = O\big(f^{-\beta/2} t^2\big).\end{aligned}$$

To sum up, the parameter λ from (64) associated with $Y^*_{v,j}$ satisfies $\lambda = O(f^{-\beta/2} t^2)$. Since $Y^*_{v,j}$ is a decreasing function, by invoking the tail inequality Theorem 20 and the estimate (41) from Lemma 18 we obtain that

$$\mathbb{P}\Big(Y^*_{v,j} \geq \mathbb{E} Y^*_{v,j} + t\Big) \leq \exp\Big(-\frac{t^2}{2\lambda}\Big) = \exp\Big(-\Omega(f^{\beta/2})\Big) = N^{-\omega(1)}.$$

For the upper tail of $Y^+_{v,j,k}$ in (63) we shall invoke the 'limited dependencies' upper tail inequality of Theorem 23 via Remark 24 (exploiting the 'good' event $\mathcal{G} = \mathcal{N}_{i+1}$). Using the notations in Theorem 23 and Remark 24, we have $\xi_a := \mathbb{1}_{\{a \in \Gamma_{i+1}\}}$ for vertices $a \in A_i$, and $Y^+_{v,j,k} = \sum_{\alpha \in \mathcal{I}} Y_\alpha$ for variables $Y_\alpha := \mathbb{1}_{\{U \subseteq \Gamma_{i+1}\}}$, where the index set $\mathcal{I}$ consists of all ordered pairs $\alpha = (U, U')$ that satisfy $U' \in Y_{v,k}(i)$ and $U \subseteq U'$, as well as $|U| = k - j$. Similarly to (59), we have

$$\mathbb{E} Y^+_{v,j,k} \leq |Y_{v,k}(i)| \cdot \binom{k}{k-j} p_i^{k-j} \leq \binom{r-1}{k} \binom{k}{k-j} \sigma^{k-j} q_i^j \pi_i^{r-1-k} D^{\frac{j}{r-1}} =: \mu. \tag{65}$$

eq:Y*vj

Next we shall bound the parameter C from Remark 24 associated with $Y^+_{v,j,k}$: to this end, given any $(U, U') \in \mathcal{I}$ with $U \subseteq \Gamma_{i+1}$, it suffices to bound the number of relevant $(\hat{U}, \hat{U}') \in \mathcal{I}$, i.e., which satisfy $U \cap \hat{U} \neq \varnothing$ and $\hat{U} \subseteq \Gamma_{i+1}$. For this we first derive one extra constraint. Since we require $\hat{U}' \in Y_{v,k}(i)$ by definition (23) of $Y_{v,k}(i)$, the k-set $\hat{U}'$ must be contained in some edge $\hat{e}$ with $v \in \hat{e}$ such that the $(r-1-k)$-set $\hat{e} \setminus (\{v\} \cup \hat{U}')$ is contained in V_i. Since we also require $\hat{U} \subseteq \Gamma_{i+1}$, for any $\hat{u} \in \hat{U}$ it then follows (using $V_{i+1} = V_i \cup \Gamma_{i+1}$) that

$$|(\hat{e} \setminus \{\hat{u}, v\}) \cap V_{i+1}| \geq |\hat{e} \setminus (\{v\} \cup \hat{U}')| + |\hat{U} \setminus \{\hat{u}\}| = (r-1-k) + (k-j-1) = r-j-2. \tag{66}$$

eq:requireme

With this extra constraint in hand, we are now ready to count the number of relevant $(\hat{U}, \hat{U}') \in \mathcal{I}$, as discussed. Namely, there are at most r ways to choose a vertex $\hat{u} \in U$ to be in the non-empty intersection $U \cap \hat{U}$. Then there are at most $\Delta_{2,r-j-2}(i+1)$ ways (see (29) for definition of this parameter) to choose an edge $\hat{e}$ that contains $\{\hat{u}, v\}$ and satisfies the constraint (66). Once $\hat{e}$ is chosen, there are at most 2^r ways to choose $\hat{U}' \subseteq \hat{e}$, and then at most 2^r ways to choose $\hat{U} \subseteq \hat{U}'$. Therefore, whenever the event $\mathcal{N}_{i+1}$ holds, using the definitions (62) and (65) of t and μ together with inequality (40) from Lemma 18 and estimate $q_i \sim q_{i+1}$ from Remark 28, it follows that the parameter C associated with $Y^+_{v,j,k}$ satisfies

$$C \leq r \cdot \Delta_{2,r-j-2}(i+1) \cdot 2^{2r} \leq r \cdot 2^{2r} \cdot D^{\frac{j}{r-1}} f^{-1} Q^{r-j-1} \leq f^{-\beta/2} \min\Big\{\frac{t^2}{\mu},\, t\Big\}. \tag{67}$$

eq:Y*vj:C

Invoking the upper tail inequality Theorem 23 via Remark 24, using (41) from Lemma 18 it follows that

$$\mathbb{P}\Big(Y^+_{v,j,k} \geq \mathbb{E} Y^+_{v,j,k} + t \text{ and } \mathcal{N}_{i+1}\Big) \leq \exp\Big(-\frac{t^2}{2C(\mu + t)}\Big) = \exp\Big(-\Omega(f^{\beta/2})\Big) = N^{-\omega(1)},$$

which completes the proof of Theorem 31 (as discussed around (63) above). □

Recall that the event $\mathcal{A}_{i+1}$ defined in (33) concerns the number $|A_{i+1}|$ of available vertices. Next we estimate the size of $|A_{i+1}|$, which together with Theorem 31 proves estimate (45) of Lemma 25. As before, here the basic proof idea is to estimate $|A_{i+1}|$ via several auxiliary random variables (see (68) and (71)–(72) below) that are more amenable to the concentration inequalities from Section 2.7.

thm:Ai **Theorem 32.** *We have* $\mathbb{P}(\neg\mathcal{A}_{i+1} \cap \mathcal{N}_{i+1} \mid \mathcal{F}_i) \le N^{-\omega(1)}$ *when* $\mathfrak{X}_{\le i}$ *holds.*

Proof. We start with the upper bound for $|A_{i+1}|$. Note that by construction

$$|A_{i+1}| \le \sum_{v \in A_i} \mathbb{1}_{\{v \notin C_{i+1}\}} =: X. \tag{68}$$

eq:def:Ai+1X

Since $\mathfrak{X}_{\le i} \subseteq \mathcal{A}_i$ implies $|A_i| \le q_i|A_0|$, using Lemma 27, we infer

$$\begin{aligned}\mathbb{E}X &\le q_i|A_0|\Big(\frac{q_{i+1}}{q_i} - \frac{1}{2}(r-1)\sigma^{1+\rho}\pi_i^{\max\{r-3,0\}}\Big)\\ &= q_{i+1}|A_0| - \underbrace{\frac{1}{2}(r-1)\sigma^{1+\rho}q_i\pi_i^{\max\{r-3,0\}}|A_0|}_{=:t}.\end{aligned} \tag{69}$$

eq:def:Ai+1y

In order to bound X from above, we now proceed similar to the upper bound on $Y^*_{v,j}$ in the proof of Theorem 31: indeed, the plan is to invoke the bounded differences inequality Theorem 20 to X, which requires us to bound the parameter λ from (57) associated with X. Note that the vertex-effect satisfies $c_u \le |Y_u(i)| \le (r-1)q_i\pi_i^{r-2}D^{\frac{1}{r-1}}$ due to $\mathfrak{X}_{\le i} \subseteq \mathcal{Y}_i$, and that Lemma 30 implies $\sum_{u\in A_i} c_u \le \sum_{u \in A_i} |Y_u(i) \cap A_i| \le (r-1)q_i\pi_i^{r-2}D^{\frac{1}{r-1}} \cdot |A_i|$. Furthermore, the stabilization-effect clearly satisfies $\hat{c_u} \le \mathbb{1}_{\{u \in A_i\}}$. Since $\mathfrak{X}_{\le i} \subseteq \mathcal{A}_i$ implies $|A_i| \le q_i|A_0|$, using (40) from Lemma 18 and the assumption $|A_0|/D^{\frac{1}{r-1}} \ge f^{\beta}$ it follows that the parameter λ from (57) satisfies (for t as defined in (69) above), say,

$$\begin{aligned}\lambda &\le p_i \cdot \Big(\max_{u \in A_i} c_u\Big) \cdot \sum_{u\in A_i} c_u + \sum_{u \in A_i} \mathbb{1}_{\{u \in A_i\}},\\ &\le \frac{\sigma}{q_i D^{\frac{1}{r-1}}} \cdot (r-1)q_i\pi_i^{r-2}D^{\frac{1}{r-1}} \cdot (r-1)q_i\pi_i^{r-2}D^{\frac{1}{r-1}}|A_i| + |A_i|\\ &\le r^2\sigma q_i^2\pi_i^{2(r-2)}D^{\frac{1}{r-1}}|A_0| = O\big(f^{-\beta/2}t^2\big).\end{aligned} \tag{70}$$

temlambda

Since X is decreasing, by the tail inequality Theorem 20 (with t as defined in (69) above) and the estimate (41) from Lemma 18 we obtain that

$$\mathbb{P}(|A_{i+1}| \ge q_{i+1}|A_0|) \le \mathbb{P}(X \ge \mathbb{E}X + t) \le \exp\Big(-\frac{t^2}{2\lambda}\Big) \le \exp\Big(-\Omega(f^{\beta/2})\Big) = N^{-\omega(1)}.$$

We now turn to the lower bound for $|A_{i+1}|$, which is more elaborate. Note that by construction

$$|A_{i+1}| \ge \underbrace{\sum_{v\in A_i} \mathbb{1}_{\{v \notin C_{i+1}\}}}_{=X} - \underbrace{\sum_{v \in A_i} \mathbb{1}_{\{v \in \Gamma_{i+1}\}}}_{=:X_1} - \sum_{2 \le j \le r-1} \underbrace{\sum_{v \in A_i} \mathbb{1}_{\{v \in C^{(j)}_{i+1}\}}}_{=:X_{2,j}}. \tag{71}$$

lowerbound

We shall enlarge the last term in the inequality (71) to make it related to X_1 in (71) and a variable we have already studied in (58). Note that if $v \in A_i$ and $v \in C^{(j)}_{i+1}$, then by definition of $C^{(j)}_{i+1}$ in (25), there exists $U \in Y_{v,j}(i)$ of size j such that $U \subseteq \Gamma_{i+1}$. Then by definition of $Y_{v,j}(i)$ in (23), there exists an edge $e \in \mathcal{H}$ such that $\{v\} \cup U \subseteq e$, $U \subseteq A_i$, and $e \setminus (\{v\} \cup U) \subseteq V_i$. For a vertex $w \in U$, since $v \in A_i$, then $W' := \{v\} \cup (U \setminus \{w\})$ is of size j and is contained in $e \cap A_i$, and $e \setminus (\{w\} \cup W') = e \setminus (\{v\} \cup U) \subseteq V_i$, therefore $W' \in Y_{w,j}(i)$ and $W := W' \setminus \{v\} \subseteq \Gamma_{i+1}$ (so $\{v\} = W' \setminus W$). It follows that

$$\begin{aligned}X_{2,j} = \sum_{v \in A_i} \mathbb{1}_{\{v \in C^{(j)}_{i+1}\}} &\le \sum_{w \in A_i} \mathbb{1}_{\{w \in \Gamma_{i+1}\}} \underbrace{\sum_{(W,W'):W' \in Y_{w,j}(i), W \subseteq W', |W| = j-1} \mathbb{1}_{\{W \subseteq \Gamma_{i+1}\}}}_{=Y^+_{w,1,j}}\\ &\le X_1 \cdot \max_{w \in A_i} Y^+_{w,1,j}.\end{aligned} \tag{72}$$

relationX3jZ

To sum up, in view of (71) and (72), to obtain the desired lower bound $|A_{i+1}| \ge \tau_{i+1}q_{i+1}|A_0|$ from (33), it remains to bound X from below, and bound X_1 and $\max_{w\in A_i} Y^+_{w,1,j}$ from above (to avoid clutter we omit the precise bounds here, and only show in the end that we indeed obtain the desired lower bound).

Bounding X defined in (68) from below is very similar to the above-discussed upper bound. Indeed, using Lemma 27 and $|A_i| \geq \tau_i q_i |A_0|$, similar to (69) here we obtain, with room to spare (using r instead $r-1$),

$$\mathbb{E}X \geq \tau_i q_{i+1}|A_0| - \underbrace{\frac{3}{2} r\tau_i \sigma^{1+\rho} q_i \pi_i^{\max\{r-3,0\}}|A_0|}_{=:3t}. \tag{73}$$

eq:lowerboun

In order to apply the bounded differences inequality from Remark 21 to X with t as defined in (73) above, we need to bound the parameter λ from (57) associated with X, for which the same asymptotic bound $\lambda = O(f^{-\beta/2}t^2)$ holds as in (70) since $\tau_i \geq 1/2$. Here we also need to bound the related parameter C from Remark 21, which by similar reasoning as for λ satisfies

$$C = \max_{u \in A_i} \max\{c_u, \hat{c}_u\} \leq \max_{u \in A_i} \max\{|Y_u(i)|, 1\} \leq (r-1) q_i \pi_i^{r-2} D^{\frac{1}{r-1}} = O(f^{-\beta/2} t).$$

Invoking Remark 21 with t as defined in (73) above, it follows that, say,

$$\begin{aligned} &\mathbb{P}\Big(X \leq \tau_i q_{i+1}|A_0| - 2r\tau_i \sigma^{1+\rho} q_i \pi_i^{\max\{r-3,0\}}|A_0|\Big) \\ &\qquad \leq \mathbb{P}(X \leq \mathbb{E}X - t) \leq \exp\Big(-\frac{t^2}{2(\lambda + Ct)}\Big) \leq \exp\Big(-\Omega(f^{\beta/2})\Big) = N^{-\omega(1)}. \end{aligned} \tag{74}$$

X1

Next we bound X_1 defined in (71) from above, for which we shall use a standard Chernoff bound (see Remark 22). Recalling $|A_i| \leq q_i|A_0|$, note that by construction

$$\mathbb{E}X_1 = |A_i| \cdot p_i \leq \frac{|A_0|\sigma}{D^{\frac{1}{r-1}}} =: \mu.$$

Similarly as before, using assumption (6) and (40) from Lemma 18, we infer that $|A_0|/D^{\frac{1}{r-1}} \geq f^{\beta}$ and thus $\mu = \Omega(f^{\beta/2})$. Applying Remark 22 with $C = 1$ and $\lambda = \mathbb{E}X_1 \leq \mu$, it follows that

$$\mathbb{P}\Big(X_1 \geq 2|A_0|\sigma/D^{\frac{1}{r-1}}\Big) \leq \mathbb{P}(X_1 \geq \mathbb{E}X_1 + \mu) \leq \exp\big(-\mu/4\big) \leq \exp\Big(-\Omega(f^{\beta/2})\Big) = N^{-\omega(1)}. \tag{75}$$

X2

With an eye on (72) we now bound $\max_{w \in A_i} Y^+_{w,1,j}$ from above, exploiting that the random variable $Y^+_{w,1,j}$ is a special case of $Y^+_{v,j,k}$ defined in (58). Similarly as in the expectation calculations for (59) and (65) for $\mathbb{E}Y^+_{v,j,k}$, note that for $w \in A_i$ and $2 \leq j \leq r-1$ (which implies $r \geq 3$) we here have

$$\begin{aligned} \mathbb{E}Y^+_{w,1,j} &\leq \binom{r-1}{j} q_i^j \pi_i^{r-1-j} D^{\frac{j}{r-1}} \cdot \binom{j}{j-1}\Big(\frac{\sigma}{q_i D^{\frac{1}{r-1}}}\Big)^{j-1} \\ &= \binom{r-1}{j}\binom{j}{j-1}\sigma^{j-1} q_i \pi_i^{r-1-j} D^{\frac{1}{r-1}} =: \mu_j. \end{aligned}$$

To bound $Y^+_{w,1,j}$ from above, we shall invoke the 'limited dependencies' upper tail inequality of Theorem 23 via Remark 24 (exploiting the 'good' event $\mathcal{G} = \mathcal{N}_{i+1}$). Whenever the event $\mathcal{N}_{i+1}$ holds, then by arguing similarly as for the corresponding bound (67) for $Y^+_{v,j,k}$, we see that the parameter C associated with $Y^+_{w,1,j}$ satisfies

$$C \leq 2^r \cdot r \cdot \Delta_{2,r-3}(i+1) \cdot 2^r \leq r \cdot 2^{2r} \cdot D^{\frac{1}{r-1}} f^{-1} Q^{r-2}.$$

Together with (40)–(41) from Lemma 18, now Remark 24 implies that

$$\begin{aligned} \mathbb{P}\Big(Y^+_{w,1,j} \geq 2\mu_j \text{ and } \mathcal{N}_{i+1}\Big) &\leq \mathbb{P}\Big(Y^+_{w,1,j} \geq \mathbb{E}Y^+_{w,1,j} + \mu_j \text{ and } \mathcal{N}_{i+1}\Big) \\ &\leq \exp\Big(-\frac{\mu_j^2}{2C \cdot 2\mu_j}\Big) \leq \exp\Big(-\Omega(f^{\beta/2})\Big) = N^{-\omega(1)}. \end{aligned}$$

Taking a union bound over all vertices $w \in A_i$ and $2 \leq j \leq r-1$, it readily follows that

$$\sum_{2 \leq j \leq r-1} \mathbb{P}\Big(\max_{w \in A_i} Y^+_{w,1,j} \geq 2\mu_j \text{ and } \mathcal{N}_{i+1}\Big) \leq N^{-\omega(1)}. \tag{76}$$

second

Finally, we are ready to bound $|A_{i+1}|$ from below. In view of the three probabilistic estimates (74)–(76), a moment's thought reveals that to this end we may assume the three bounds

$$X \geq \tau_i q_{i+1}|A_0| - 2r\tau_i\sigma^{1+\rho}q_i\pi_i^{\max\{r-3,0\}}|A_0|, \tag{77}$$
eq:X:lower

$$X_1 \leq 2|A_0|\sigma/D^{\frac{1}{r-1}}, \tag{78}$$
eq:X1:upper

$$\max_{w\in A_i} Y^+_{w,1,j} \leq 2\mu_j \quad \text{for all } 2 \leq j \leq r-1. \tag{79}$$
eq:Yw1j:uppe

Turning to the details, using the inequalities $\pi_i \geq \sigma$, $\tau_i \geq 1/2$ and $\rho < 1$, for $2 \leq j \leq r-1$ we first note that

$$2\mu_j = 2\binom{r-1}{j}\binom{j}{j-1}\sigma^{j-1}q_i\pi_i^{r-1-j}D^{\frac{1}{r-1}} \ll \tau_i\sigma^{\rho}q_i\pi_i^{\max\{r-3,0\}}D^{\frac{1}{r-1}}.$$

Using inequality (72), the above-mentioned upper bounds (78)–(79) for X_1 and $Y^+_{w,1,j}$ thus imply that

$$\sum_{2\leq j\leq r-1} X_{2,j} \leq \sum_{2\leq j\leq r-1} 2\frac{|A_0|\sigma}{D^{\frac{1}{r-1}}}\cdot 2\mu_j \leq \tau_i\sigma^{1+\rho}q_i\pi_i^{\max\{r-3,0\}}|A_0|.$$

Using inequality (71), the above-mentioned bounds (77)–(78) for X and X_1 therefore yield that, say,

$$|A_{i+1}| \geq \tau_i q_{i+1}|A_0| - 4r\tau_i\sigma^{1+\rho}q_i\pi_i^{\max\{r-3,0\}}|A_0|.$$

Since $q_i/q_{i+1} \leq 5/4$ by Remark 28, using $\pi_i \leq \pi_m$ and $\tau_i \leq 1$ it follows that

$$|A_{i+1}| \geq q_{i+1}|A_0|\Big(\tau_i - 5r\sigma^{\rho}\pi_m^{\max\{r-2,1\}}\frac{\sigma}{\pi_m}\Big).$$

Note that (39) in Lemma 18 implies $\sigma^{\rho}\pi_i^{\max\{r-2,1\}} \ll (\log f)^{-1}$. Using $\pi_{i+1} = \pi_i + \sigma$, see (18), we also have $\tau_{i+1} = \tau_i - (\log f)^{-1}\cdot\sigma/\pi_m$. Putting these estimates together, we obtain that

$$|A_{i+1}| \geq \tau_{i+1}q_{i+1}|A_0|,$$

which in view of (33) completes the proof of Theorem 32, as discussed. □

sub:II

3.2 Size of the final independent set: event $\mathcal{I}_m$

In this subsection we prove estimate (47) via Theorem 33 below, which deals with the size of the final independent set I_m via the event $\mathcal{I}_m$ defined in (31). Inspired by [15, Section 3.5], in order to estimate $|I_m|$ it will be convenient to think of the entire nibble construction as one evolving stochastic process, see the proof of Lemma 34 below (this viewpoint differs from the previous Section 3.1, where we analyzed the relevant random variables by separately considering the changes in each 'step' of the nibble construction).

thm:II **Theorem 33.** *We have* $\mathbb{P}(\neg\mathcal{I}_m \cap \mathfrak{X}_{\leq m}) \leq N^{-\omega(1)}$.

boundofXY **Lemma 34.** *Let $\mathcal{T}$ be the event that the following bounds hold:*

$$X := \sum_{0\leq i<m} |A_i \cap \Gamma_{i+1}| \in \big[(1-\delta/2)\mu^-,\ (1+\delta/2)\mu^+\big],$$

$$Y := \sum_{0\leq i<m} |A_i \cap V(\mathcal{D}_{i+1})| \leq \delta^2\mu^-/9,$$

where $\mu^+ = \sum_{0\leq i<m}\lfloor q_i|A_0|\rfloor p_i$ *and* $\mu^- = \sum_{0\leq i<m}\lceil\tau_i q_i|A_0|\rceil p_i$*. Then* $\mathbb{P}(\neg\mathcal{T}\cap\mathfrak{X}_{\leq m}) \leq N^{-\omega(1)}$.

Proof of Theorem 33 (assuming Lemma 34). It suffices to show that the event $\mathcal{T}$ from Lemma 34 implies the event $\mathcal{I}_m$ defined in (31). To this end we may assume that $\delta \leq 1$ holds (since otherwise the target bound (31) for $|\mathcal{I}_m|$ holds trivially). Turning to the details, by the recursive definition of I_m we have

$$X - Y \leq |I_m| \leq X. \tag{80}$$
eq:FI0

Noting $\mu^- \geq \tau_m \mu^+ \geq (1-\delta/2)\mu^+$, the event $\mathcal{T}$ implies $X \leq (1+\delta/2)\mu^+$ and

$$X - Y \geq (1 - \delta/2 - \delta^2/9) \cdot \mu^- \geq (1 - \delta + \delta^2/8)\mu^+.$$

Recalling the target bound (31) for $|\mathcal{I}_m|$, it therefore is enough to show that $\mu^+ \sim \xi|A_0|\big(\log f)/D\big)^{1/(r-1)}$. But this is straightforward: using $q_i|A_0|p_i = |A_0|\sigma/D^{\frac{1}{r-1}} \geq \sigma f^\beta \gg p_i = \sigma/(q_i D^{\frac{1}{r-1}})$ by the assumed bound (6) and estimate (40) from Lemma 18, we obtain that

$$\mu^+ = \sum_{0\leq i<m} (q_i|A_0| \pm 1)p_i \sim \sum_{0\leq i<m} q_i|A_0| \frac{\sigma}{q_i D^{\frac{1}{r-1}}} = |A_0| \frac{\sigma m}{D^{\frac{1}{r-1}}} \sim \xi |A_0| \Big(\frac{\log f}{D}\Big)^{\frac{1}{r-1}}, \tag{81}$$

eq:mup

completing the proof of Theorem 33 (modulo the proof of Lemma 34 given below). □

Proof of Lemma 34. We start with estimating $X = \sum_{0\leq i<m} |A_i \cap \Gamma_{i+1}|$. Heuristically speaking, here the basic idea is that if $\mathfrak{X}_i \subseteq \mathcal{A}_i$ holds, then the fact that $|A_i|$ is an integer implies $\lceil \tau_i q_i |A_0| \rceil \leq |A_i| \leq \lfloor q_i|A_0| \rfloor$. Since Γ_{i+1} is a p_i-random subset of A_i, we can therefore stochastically dominate X from below and above by $\sum_{0\leq i<m} \mathrm{Bin}(\lceil \tau_i q_i|A_0|\rceil, p_i)$ and $\sum_{0\leq i<m} \mathrm{Bin}(\lfloor q_i|A_0|\rfloor, p_i)$, which in turn can both be estimated via suitable Chernoff's bounds. To make this rigorous, we set

$$X^+_{i+1} := \mathbb{1}_{\{\mathfrak{X}_i\}} \sum_{v\in A_i} \mathbb{1}_{\{v\in\Gamma_{i+1}\}} \qquad \text{and} \qquad X^+ := \sum_{0\leq i<m} X^+_{i+1}.$$

Note that $X = X^+$ when $\mathfrak{X}_{\leq m} = \bigcap_{0\leq i\leq m} \mathfrak{X}_i$ holds. Let $Z^+_{i+1} \overset{\mathrm{d}}{=} \mathrm{Bin}(\lfloor q_i|A_0|\rfloor,\, p_i)$ be independent random variables (where $\overset{\mathrm{d}}{=}$ means equality in distribution, as usual). Since the $\mathcal{F}_i$-measurable event $\mathfrak{X}_i \subseteq \mathcal{A}_i$ implies $|A_i| \leq q_i|A_0|$, it is not hard to see that $\mathbb{P}(X^+_{i+1} \geq t \mid \mathcal{F}_i) \leq \mathbb{P}(Z^+_{i+1} \geq t)$ for $t \in \mathbb{R}$. Setting

$$Z^+ := \sum_{0\leq i<m} Z^+_{i+1} \overset{\mathrm{d}}{=} \sum_{0\leq i<m} \mathrm{Bin}\Big(\lfloor q_i|A_0|\rfloor,\, p_i\Big), \tag{82}$$

eq:Zplus

a standard stochastic domination argument shows that $\mathbb{P}(X^+ \geq t) \leq \mathbb{P}(Z^+ \geq t)$ for $t \in \mathbb{R}$, so that

$$\mathbb{P}(X \geq t \text{ and } \mathfrak{X}_{\leq m}) \leq \mathbb{P}(X^+ \geq t) \leq \mathbb{P}(Z^+ \geq t). \tag{83}$$

eq:XUT

Since $\mathfrak{X}_i \subseteq \mathcal{A}_i$ also implies $|A_i| \geq \tau_i q_i |A_0|$, an analogous argument shows

$$\mathbb{P}(X \leq t \text{ and } \mathfrak{X}_{\leq m}) \leq \mathbb{P}(Z^- \leq t) \qquad \text{with} \qquad Z^- \overset{\mathrm{d}}{=} \sum_{0\leq i<m} \mathrm{Bin}\Big(\lceil \tau_i q_i |A_0|\rceil,\, p_i\Big). \tag{84}$$

eq:XLT

Using estimate (81) together with $\mu^- \geq \frac{1}{2}\mu^+$ and $\delta^2 \frac{|A_0|}{D^{\frac{1}{r-1}}} \geq f^{\beta/2}$ (using that $|A_0|/D^{\frac{1}{r-1}} \geq f^\beta$ by (6), and that $\delta \geq (\log f)^{-1}$ by the assumptions of Theorem 16) as well as $\mathbb{E}Z^\pm = \mu^\pm$, by standard Chernoff bounds (see Remark 22) and estimate (41) from Lemma 18 it follows that

$$\begin{aligned}\mathbb{P}\Big(X \notin [(1-\delta/2)\mu^-, (1+\delta/2)\mu^+] \text{ and } \mathfrak{X}_{\leq m}\Big) &\leq \mathbb{P}\Big(Z^- \leq (1-\delta/2)\mu^-\Big) + \mathbb{P}\Big(Z^+ \geq (1+\delta/2)\mu^+\Big) \\ &\leq \exp(-\delta^2\mu^-/8) + \exp(-\delta^2\mu^+/12) \\ &\leq \exp\Big(-\Omega(f^{\beta/2})\Big) = N^{-\omega(1)}.\end{aligned} \tag{85}$$

eq:XUTLYmu

Finally, turning to $Y = \sum_{0\leq i<m} |A_i \cap \mathcal{D}_{i+1}|$, for brevity we define

$$Y_{i+1} := |A_i \cap \mathcal{D}_{i+1}| \qquad \text{and} \qquad y := \delta^2\mu^-/9. \tag{86}$$

def:Yi1:y

Note that $Y = \sum_{0\leq i<m} Y_{i+1}$ and $Y_{i+1} \in \mathbb{N}$. Since $\mathfrak{X}_{\leq i} = \bigcap_{0\leq j\leq i} \mathfrak{X}_j$, a union bound argument gives

$$\begin{aligned}\mathbb{P}(Y \geq \delta^2\mu^-/9 \text{ and } \mathfrak{X}_{\leq m}) &\leq \sum_{\substack{(y_1,\ldots,y_m)\in\mathbb{N}^m \\ \sum_{0\leq i<m} y_{i+1} = \lceil y\rceil}} \mathbb{P}\Big(\bigcap_{0\leq i<m} (Y_{i+1} \geq y_{i+1} \text{ and } \mathfrak{X}_{\leq i+1})\Big) \\ &\leq \sum_{\substack{(y_1,\ldots,y_m)\in\mathbb{N}^m \\ \sum_{0\leq i<m} y_{i+1} = \lceil y\rceil}} \prod_{0\leq i<m} \mathbb{P}\Big(Y_{i+1} \geq y_{i+1} \mid \bigcap_{0\leq j<i} (Y_{j+1} \geq y_{j+1} \text{ and } \mathfrak{X}_{\leq j+1})\Big).\end{aligned} \tag{87}$$

eq:sumbound

Gearing up to apply the 'limited dependencies' upper tail inequality Theorem 23 to Y_{i+1}, in view of $\mathcal{D}_{i+1} \subseteq B_{i+1}$ and $I_i \subseteq V_i$ we define (recall that we identify the index $\alpha \in \mathcal{I}$ with its first coordinate α_1 in this application) the index set

$$\mathcal{I} := \bigcup_{2 \le j \le r} \Big\{ W \subseteq A_i : |W| = j, \text{ and there exists } U \subseteq V_i, W \cup U \in E_{\mathcal{H}} \Big\}.$$

For $\alpha \in \mathcal{I}$, there exists a $v \in \alpha$ and $v \in A_i$, $\alpha \setminus \{v\} \in Y_{v,|\alpha|-1}$, we obtain by the usual reasoning (using $\pi_i \ge \sigma$, and that $\mathfrak{X}_i \subseteq \mathcal{A}_i \cap \mathcal{Y}_i$ implies the estimates $|A_i| \le q_i|A_0|$ and $|Y_{v,j}(i)| \le \binom{r-1}{j} q_i^j \pi_i^{r-1-j} D^{\frac{j}{r-1}}$) that

$$\begin{aligned}
\sum_{\alpha \in \mathcal{I}} \mathbb{E}(\mathbb{1}_{\{\alpha \subseteq \Gamma_{i+1}\}} \mid \mathcal{F}_i) &\le \sum_{v \in A_i} \sum_{\alpha \in \mathcal{I}: v \in \alpha} p_i^{|\alpha|} \le \sum_{v \in A_i} \sum_{1 \le j \le r-1} \sum_{W \in Y_{v,j}(i)} p_i^{|W \cup \{v\}|} \\
&\le q_i |A_0| \sum_{1 \le j \le r-1} \binom{r-1}{j} q_i^j \pi_i^{r-1-j} D^{\frac{j}{r-1}} \cdot \Big(\frac{\sigma}{q_i D^{\frac{1}{r-1}}} \Big)^{j+1} \\
&\le \frac{|A_0|}{D^{\frac{1}{r-1}}} \sum_{1 \le j \le r-1} \binom{r-1}{j} \sigma^{j+1} \pi_i^{r-1-j} \\
&\le \frac{|A_0|}{D^{\frac{1}{r-1}}} r \cdot 2^r \sigma^2 \pi_i^{r-2} =: \mu^*_{i+1}.
\end{aligned}$$

Since $\mathcal{D}_{i+1}$ is a collection of vertex-disjoint elements of B_{i+1} (and thus $\{\hat{\alpha} \in \mathcal{D}_{i+1} : \alpha \cap \hat{\alpha} \neq \varnothing\} = \{\alpha\}$ for all $\alpha \in \mathcal{D}_{i+1}$), using $V(\mathcal{D}_{i+1}) = \bigcup_{\alpha \in \mathcal{D}_{i+1}} \alpha \subseteq \Gamma_{i+1} \subseteq A_i$, $|\alpha| \le r$ and $I_i \subseteq V_i$, it is easy to see that

$$Y_{i+1} = \sum_{\alpha \in \mathcal{D}_{i+1}} |\alpha \cap A_i| \le r \cdot \sum_{\alpha \in \mathcal{I} \cap \mathcal{D}_{i+1}} \mathbb{1}_{\{\alpha \subseteq \Gamma_{i+1}\}} \le r Z_1,$$

where Z_1 is defined as in Theorem 23. Applying the upper tail inequality (44) with $C = 1$ and $\mu = \mu^*_{i+1}$ (in the probability space conditional on $\mathcal{F}_i$), whenever $\mathfrak{X}_{\le i}$ holds it follows that

$$\mathbb{P}(Y_{i+1} \ge y_{i+1} \mid \mathcal{F}_i) \le \mathbb{P}(Z_1 \ge y_{i+1}/r \mid \mathcal{F}_i) \le \begin{cases} \Big(\frac{e\mu^*_{i+1}}{y_{i+1}/r} \Big)^{y_{i+1}/r} \le \sigma^{y_{i+1}/(2r)} & \text{if } y_{i+1} \ge \frac{re\mu^*_{i+1}}{\sqrt{\sigma}}, \\ 1 & \text{otherwise.} \end{cases} \tag{88}$$

Boundeachter

Comparing the definition of $\sum_{0 \le i < m} \mu^*_{i+1}$ with μ^-, using $\tau_i \ge \tau_m \ge \frac{1}{2}$ together with the estimate $\sqrt{\sigma}\pi_i^{r-2} \ll (\log f)^{-2} \le \delta^2$ (that is implied by (39) from Lemma 18), it follows that

$$\sum_{\substack{0 \le i < m: \\ y_{i+1} \le re\mu^*_{i+1}/\sqrt{\sigma}}} y_{i+1} \le \frac{re}{\sqrt{\sigma}} \sum_{0 \le i < m} \mu^*_{i+1} \ll \frac{\delta^2 \mu^-}{9} = y.$$

So, inserting (88) into (87), we infer that

$$\mathbb{P}(Y \ge \delta^2 \mu^-/9 \text{ and } \mathfrak{X}_{\le m}) \le \sum_{\substack{(y_1, \dots, y_m) \in \mathbb{N}^m \\ \sum_{0 \le i < m} y_{i+1} = \lceil y \rceil}} \sigma^{\frac{\lceil y \rceil}{2r} - o(y)} \le (y+2)^m \sigma^{\frac{\lceil y \rceil}{4r}}, \tag{89}$$

eq:unionboun

and so it remains to estimate the right-hand side of (89). Recalling the definition (86) of the parameter y, by combining the assumed bound $|A_0|/D^{\frac{1}{r-1}} = N/D^{\frac{1}{r-1}} \ge f^\beta$ from (6) with the assumption $\delta \ge (\log f)^{-1}$ from Theorem 16, it follows that

$$y = \Theta\Big(\delta^2 |A_0| \big(\frac{\log f}{D} \big)^{\frac{1}{r-1}} \Big) \ge f^{2\beta/3}.$$

Furthermore, using estimates (40)–(41) from Lemma 18 we also see that

$$\log(y+2) \le \Theta\Big(\log n \Big) = O(f^{\beta/2}),$$

which together with $m = (\log f)^{O(1)}$ and $\sigma = (\log f)^{-9r}$ implies that

$$m\log(y+2) + \frac{y}{4r}\log\sigma \;\le\; (\log f)^{O(1)} \cdot O(f^{\beta/2}) - f^{2\beta/3}\cdot\Theta(\log\log f) \;\ll\; -f^{\beta/2}.$$

Exponentiating both sides, using (89) it thus follows that

$$\mathbb{P}(Y \ge \delta^2\mu^-/9 \text{ and } \mathfrak{X}_{\le m}) \le (y+2)^m \sigma^{\frac{\lfloor y \rfloor}{4r}} \le \exp\Big(-f^{\beta/2}\Big) = N^{-\omega(1)},$$

which together with estimate (85) completes the proof of Lemma 34. □

:Crudebounds

3.3 Crude auxiliary bounds: events $\mathcal{N}_i$ and $\mathcal{P}_i$

In this subsection we prove estimate (46) via Theorem 35 below, which concerns with the crude (density based) bounds of the auxiliary events $\mathcal{N}_i$ and $\mathcal{P}_i$ defined in (35)–(36).

:probofCrude **Theorem 35.** *We have* $\mathbb{P}\Big(\neg\big(\mathcal{N}_i \cap \mathcal{P}_i\big)$ *for some* $0 \le i \le m\Big) \le N^{-\omega(1)}$.

In contrast to the previous Sections 3.1–3.2, in the upcoming proof of Theorem 35 we will not analyze the finer details of the nibble construction: instead we will exploit a crude bound on V_i that follows from stochastic domination, which conveniently allows us to work with independent vertex-choices. Turing to the details, let V^+ be a P-random vertex-subset of $A_0 = V_{\mathcal{H}}$, where each vertex $v \in A_0 = V_{\mathcal{H}}$ is included independently with probability $P = D^{-\frac{1}{r-1}}Q$, where $Q = f^{2\xi^{r-1}}$ as defined in (30). Since $q_i \ge q_m \ge f^{-\xi^{r-1}}$, we deterministically have

$$\sum_{0 \le i < m} p_i = \sum_{0 \le i < m} \frac{\sigma}{q_i D^{\frac{1}{r-1}}} \le \frac{\sigma m f^{\xi^{r-1}}}{D^{\frac{1}{r-1}}} \le \frac{\xi(\log f)^{\frac{1}{r-1}} f^{\xi^{r-1}}}{D^{\frac{1}{r-1}}} \le P. \tag{90}$$ eq:coupling:

For every $0 \le i \le m$, by construction of the vertex-set V_i (see Section 2.2) it follows that there is a natural coupling satisfying

$$V_i \subseteq V^+. \tag{91}$$ eq:coupling

With an eye on the definition (29) of $\Delta_{J,\ell}$ and definition (35) of $\mathcal{N}_i$, note that when (91) holds we have

$$\Delta_{J,\ell}(i) \le \sum_{e \in E_{\mathcal{H}}: J \subseteq e} \mathbb{1}_{\{|(e\setminus J)\cap E^+| \ge \ell\}} \le \sum_{e \in E_{\mathcal{H}}: J \subseteq e} \;\sum_{W \subseteq e\setminus J: |W| = \ell} \mathbb{1}_{\{W \subseteq E^+\}} =: \Delta^+_{J,\ell}. \tag{92}$$ eq:def:Delta

For later reference, we now introduce the following auxiliary event

$$\mathcal{N}^+ := \Big\{\Delta^+_{j,\ell} := \max_{J \subseteq A_0: |J| = j} \Delta^+_{J,\ell} \le D^{\frac{r-(j+\ell)}{r-1}} f^{-\mathbb{1}_{\{j+\ell<r\}}} Q^{\ell+1} \text{ for all } j,\ell \in \mathbb{N} \text{ with } 2 \le j \le r-\ell\Big\}. \tag{93}$$ eq:def:Delta

lemma:crude1 **Lemma 36.** *We have* $\mathbb{P}(\neg\mathcal{N}^+) \le N^{-\omega(1)}$.

Proof. We shall use the moment method, inspired by [22, 10]. Fix $j,\ell \in \mathbb{N}$ and $J \subseteq A_0$ with $|J| = j$ and $2 \le j \le r - \ell$. Assume that $\kappa \in \mathbb{Z}^+$ satisfies $1 \le \kappa \le \lceil Q/\ell \rceil$. By a similar argument as in Section 2.5 (there are at most Δ_j ways to choose an edge e containing J, and at most $\binom{r-j}{\ell}$ ways to choose the corresponding $W \subseteq e \setminus J$) we obtain

$$\mathbb{E}\Delta^+_{J,\ell} \le \Delta_j \binom{r-j}{\ell} P^\ell \le r^\ell D^{\frac{r-(j+\ell)}{r-1}} f^{-\mathbb{1}_{\{j+\ell<r\}}} Q^\ell =: \mu_{j,\ell}. \tag{94}$$ eq:EDeltaar-

For brevity, we henceforth write $\Delta^+_{J,\ell} = \sum_{(W,e)} \mathbb{1}_{\{W \subseteq V^+\}}$, see (92). Note that for $\kappa \ge 2$ we have

$$\mathbb{E}(\Delta^+_{J,\ell})^\kappa = \sum_{(W_1,e_1),\dots,(W_{\kappa-1},e_{\kappa-1})} P^{|\cup_{i=1}^{\kappa-1} W_i|} \sum_{(W_\kappa,e_\kappa)} P^{|W_\kappa \setminus \cup_{i=1}^{\kappa-1} W_i|}. \tag{95}$$ eq:powerofED

Taking different sizes of overlaps of W_κ with $\cup_{i=1}^{\kappa-1} W_i$ into account, by mimicking (94) it follows that

$$\sum_{(W_\kappa, e_\kappa)} P^{|W_\kappa \setminus \cup_{i=1}^{\kappa-1} W_i|} \le \mu_{j,\ell} + \sum_{1 \le k \le \ell} \binom{(\kappa-1)\ell}{k} \Delta_{j+k} \binom{r-j-k}{\ell-k} P^{\ell-k}. \qquad (96)$$

eq:overlaps1

Note that $\kappa - 1 \le Q/\ell$ implies $\binom{(\kappa-1)\ell}{k} \le Q^k$. Furthermore, using $\Delta_{j+k} \le D^{\frac{r-(j+k)}{r-1}} f^{-\mathbb{1}_{\{j+\ell<r\}}}$ and $Q \gg 1$, the right-handed side of (96) is at most $\mu_{j,\ell} Q^{1/4}$, with room to spare. By induction, we infer

$$\mathbb{E}(\Delta^+_{J,\ell})^\kappa \le \big(\mu_{j,\ell} Q^{1/4}\big)^\kappa.$$

Setting $\kappa := \lceil Q/\ell \rceil$, using Markov's inequality and estimate (41) from Lemma 18 we obtain

$$\mathbb{P}(\Delta^+_{J,\ell} \ge \mu_{j,\ell} Q^{1/2}) \le \frac{V(\Delta^+_{J,\ell})^\kappa}{(\mu_{j,\ell} Q^{1/2})^\kappa} \le Q^{-\kappa/4} \le \exp\big(-\Omega(Q)\big) = N^{-\omega(1)},$$

which in view of $r^\ell Q^{\ell+1/2} \ll Q^{\ell+1}$ completes the proof by taking a union bound over all j, ℓ and J. □

We next focus on the variable $\sum_{J \in Y_{v,j}(i)} |Y_w(i) \cap J|$ for distinct vertices v, w. By definition (23) of $Y_{u,k}(i)$, any vertex-set $U \in Y_{u,k}(i)$ has the following two properties: $|U| = k$, and there exists an edge e satisfying $u \in e$, $U \subseteq e \setminus \{u\}$, and $e \setminus (\{u\} \cup U) \subseteq V_i$. In the upcoming[3] definition (98) of the family $\mathcal{I}(v,w,j)$ of indices, we shall view the set B_v as union of $\{v\}$ with some vertex-set of $Y_{v,j}(i)$, and the B_w as union of $\{w\}$ with some vertex-set of $Y_w(i)$; therefore, by relaxing the requirement that certain vertices are in A_i (i.e., available), when the coupling (91) holds it follows that

$$\sum_{J \in Y_{v,j}(i)} |Y_w(i) \cap J| \le \sum_{(\alpha_1,(e,e')) \in \mathcal{I}(v,w,j)} \mathbb{1}_{\{\alpha_1 \subseteq V^+\}} =: Y^+_{v,w,j}, \qquad (97)$$

def:Gammaaa'

where for distinct vertices $v, w \in A_0$ and $1 \le j \le r-1$ we used

$$\begin{aligned} \mathcal{I}(v,w,j) := \big\{(\alpha_1,(e,e')) \ : \ & e, e' \in E_{\mathcal{H}},\ v \in e,\ w \in e', \text{ and there exist } B_v \subseteq e, B_w \subseteq e' \\ & \text{with } v \in B_v,\ |B_v| = j+1,\ w \in B_w,\ |B_w| = 2, \\ & |(B_v \setminus \{w\}) \cap (B_w \setminus \{v\})| = 1, \text{ and } \alpha_1 = (e \setminus B_v) \cup (e' \setminus B_w)\big\}. \end{aligned} \qquad (98)$$

eq:def:index

With this in mind, we now introduce the following auxiliary event

$$\mathcal{P}^+ := \Big\{ Y^+_{v,w,j} \le D^{\frac{j}{r-1}} f^{-1} Q^{2r-(j+1)} \text{ for all distinct } v, w \in A_0 \text{ and } 1 \le j \le r-1 \Big\}. \qquad (99)$$

eq:def:event

lemma:crude2 **Lemma 37.** *We have* $\mathbb{P}(\neg \mathcal{P}^+ \cap \mathcal{N}^+) \le N^{-\omega(1)}$.

Proof. For $r = 2$, we note that (99) holds deterministically since $Y^+_{v,w,1} \le \Gamma(\mathcal{H}) \le D f^{-1}$. So we may henceforth assume $r \ge 3$. Here the plan is to invoke the 'limited dependencies' upper tail inequality Theorem 23 via Remark 24 (exploiting the 'good' event $\mathcal{G} = \mathcal{N}^+$). To estimate $\mathbb{E} Y^+_{v,w,j}$, we will increase the cardinality of the index family $\mathcal{I}$ by considering the family $\tilde{\mathcal{I}}$ of tuples (B_v, B_w, e, e') that satisfy the requirement in (98). (We overcount because different tuples in $\tilde{\mathcal{I}}$ can correspond to same index in $\mathcal{I}$.) Then

$$\mathbb{E} Y^+_{v,w,j} \le \sum_{(B_v, B_w, e, e') \in \tilde{\mathcal{I}}} P^{|(e \setminus B_v) \cup (e' \setminus B_w)|}. \qquad (100)$$

eq:enlargede

To estimate (100), for fixed distinct vertices $v, w \in A_0$, we first assume that $j \ge 2$. Note that there are at most D ways to choose an edge e containing v. Then there are at most $2^r \cdot 2^r = 2^{2r}$ ways to choose vertex-sets B_v and $S := (B_v \setminus \{w\}) \cap (B_w \setminus \{v\})$ (with B_w to be determined later) in e such that $|B_v| = j+1$ and $|S| = 1$ as required in (98). Then from e, there are at most 2^r ways to determine the intersection $U := (e \setminus B_v) \cap (e' \setminus B_w)$ (with e', B_w to be determined later), which is supposed to be in V^+. Assume

[3] The reason for using the set B_v in the definition (98) of the family $\mathcal{I}(v,w,j)$ is to include the case that w is contained in some vertex-set of $Y_{v,j}(i)$; the reason for using the set B_w is similar.

$|U| =: \ell$, then $\ell \leq |e \setminus B_v| = r-(j+1)$. Noting that $\{w\}, S, U$ are disjoint, one can find at most $\Delta_{2+\ell}$ edges e' containing the $(2+\ell)$-set $\{w\} \cup S \cup U$. Finally there are at most 2^r ways to determine B_w in e'. At this point, we have determined all of B_v, B_w, e, e'. With an eye on (100), noting that

$$|(e \setminus B_v) \cup (e' \setminus B_w)| = |e \setminus B_v| + |e' \setminus B_w| - |U| = r-(j+1)+r-2-\ell,$$

then using $r-(j+1) \leq r-3$ to infer $\Delta_{2+\ell} \leq D^{\frac{r-(2+\ell)}{r-1}} f^{-1}$ from (4), it follows that

$$\begin{aligned}\mathbb{E}Y^+_{v,w,j} &\leq \sum_{0 \leq \ell \leq r-(j+1)} D \cdot 2^{2r} \cdot 2^r \cdot \Delta_{2+\ell} \cdot 2 \cdot P^{r-(j+1)+r-2-\ell} \\ &\leq r \cdot 2^{4r} D^{\frac{j}{r-1}} f^{-1} Q^{2r-(j+1)-2} =: \mu_j.\end{aligned} \tag{101}$$

eq:muaa'k

We now bound $\mathbb{E}Y^+_{v,w,j}$ in the remaining case $j=1$. By analogous reasoning, contributions of the cases where $|U| = \ell \leq r-3$ can again be bounded as in inequality (101) above. In the exceptional case $\ell = r-2$, since we require $|S| = 1$, all possible pairs of edges e, e' must satisfy $v \in e \setminus e'$, $w \in e' \setminus e$, and $|e \cap e'| = r-1$. This number of pairs can thus be bounded by $\Gamma(\mathcal{H})$ defined in (2). So after determining B_v, B_w in e, e' respectively in at most 2^{2r} ways, their contribution to the expectation can be bounded by $2^{2r} \cdot \Gamma(\mathcal{H}) P^{r-2} \leq 2^{2r} D^{1/(r-1)} f^{-1} Q^{r-2}$. Adding up all these contributions, it again follows that $\mathbb{E}Y^+_{v,w,j} \leq \mu_j$ holds, as before.

Next we bound the parameter C from Remark 24 associated with $Y^+_{v,w,j}$: given any $(\alpha_1, (e,e')) \in \mathcal{I}$ with $\alpha_1 \in V^+$, it suffices to bound the number of relevant $(\hat{\alpha}_1, (\hat{e}, \hat{e}')) \in \mathcal{I}$, i.e., which satisfy $\alpha_1 \cap \hat{\alpha}_1 \neq \varnothing$ and $\hat{\alpha}_1 \subseteq V^+$. To this end, note that there are at most $2r$ ways to choose a vertex $\hat{u} \in \alpha_1$ to be in the non-empty intersection $\alpha_1 \cap \hat{\alpha}_1$. Inspecting the constraints in (102), we infer that $\hat{u} \in (\hat{e} \setminus \{v\}) \cup (\hat{e}' \setminus \{w\})$ must hold. For the sake of argument, let us first count the number of choices in the case $\hat{u} \in \hat{e} \setminus \{v\}$: then by inspecting (98) we infer (noting that in this case $\hat{u} \in \hat{e} \setminus B_v$ most hold) that $j+1 < r$ and

$$|(\hat{e} \setminus \{v, \hat{u}\}) \cap V^+| \geq r-(j+1)-1, \tag{102}$$

eq:hatevusiz

so that there are at most $\Delta^+_{2,r-(j+1)-1}$ ways (see (92)–(93) for definition of this parameter) to choose an edge $\hat{e}$ that contains $\{\hat{u}, v\}$ and satisfies the constraints in (102). Once $\hat{e}$ is chosen, there are at most 2^r ways to choose a vertex $\tilde{w}$ in the size-1 intersection appearing in (98), and then there are at most $\Delta^+_{2,r-2}$ ways ways to choose an edge $\hat{e}'$ that contains $\{\tilde{w}, w\}$ and satisfies the constraints in (102). Once $\hat{e}$ and $\hat{e}'$ are chosen, there are at most 2^{2r} ways to choose $\hat{\alpha}_1 \subseteq \hat{e} \cup \hat{e}'$. The counting argument in the case $\hat{u} \in \hat{e}' \setminus \{w\}$ is similar, with the difference that in this case $r > 3$ and $|(\hat{e}' \setminus \{w, \hat{u}\}) \cap V^+| \geq r-2-1 = r-3$ hold (so that in the above argument now $\Delta^+_{2,r-(j+1)-1}$ and $\Delta^+_{2,r-2}$ need to be replaced by $\Delta^+_{2,r-3}$ and $\Delta_{2,r-(j+1)}$, respectively). Therefore, whenever the event $\mathcal{N}^+$ holds, it follows that the parameter C associated with $Y^+_{v,w,j}$ satisfies

$$\begin{aligned}C &\leq 2r \cdot \Big(\mathbb{1}_{\{j+1<r\}} \Delta^+_{2,r-j-2} \cdot 2^r \cdot \Delta^+_{2,r-2} \cdot 2^{2r} + \mathbb{1}_{\{r>3\}} \Delta^+_{2,r-3} \cdot 2^r \cdot \Delta^+_{2,r-(j+1)} \cdot 2^{2r} \Big) \\ &\leq r \cdot 2^{4r} D^{\frac{j}{r-1}} f^{-1} Q^{2r-(j+1)-1} =: C_j.\end{aligned}$$

Finally, with an eye on the auxiliary event $\mathcal{P}^+$ defined in (99), set $z_j := D^{\frac{j}{r-1}} f^{-1} Q^{2r-(j+1)}$. Invoking the upper tail inequality Theorem 23 via Remark 24, using (41) from Lemma 18 it follows that

$$\mathbb{P}\Big(Y^+_{v,w,j} \geq z_j \text{ and } \mathcal{N}^+\Big) \leq \Big(\frac{e \cdot \mu_j}{z_j}\Big)^{\frac{z_j}{C_j}} \leq \exp\big(-\Omega(Q)\big) = N^{-\omega(1)}.$$

This completes the proof by taking a union bound over all possible vertex pairs (v,w) and values of j. □

Finally, by combining the coupling from (91) with inequalities (92) and (97), it follows that Lemmas 36–37 establish Theorem 35, completing the proof of Lemma 25 (and thus Theorem 16, which in turn implies Theorem 2, as discussed).

3.4 Strengthening of Theorem 2: proof of Corollary 3

randomsubset

In this subsection we exploit that our semi-random construction only samples rather few random vertices from $V = V_{\mathcal{H}}$. Indeed, all vertices ever sampled are contained in $V_m \subseteq V$, which by construction is much smaller than V. More importantly, by the coupling from (91) we know that V_m is stochastically contained in a P-random vertex-subset of V, which together with $I \subseteq V_m$ this then allows us to conclude (via a coupling argument) that the resulting independent set I is also contained in such a P-random vertex-subset of V.

Proof of Corollary 3. For concreteness we set $\xi := \min\big\{(\frac{\beta}{37})^{\frac{1}{r-1}}, (\frac{\zeta}{2})^{\frac{1}{r-1}}\big\}$ in the above proof of Theorem 16 (and thus Theorem 2), where the first requirement comes from the discussion around (38) and the second ensures $f^{\zeta} \geq f^{2\xi^{r-1}} = Q$ in (30). Using the coupling from (91) we see that $I \subseteq V_m \subseteq V^+$, where V^+ is a P-random subset of $A_0 = V_{\mathcal{H}}$ by the definition above (90). This readily establishes Corollary 3 in view of the definition (30) of $P = D^{-\frac{1}{r-1}}Q \leq D^{-\frac{1}{r-1}}f^{\zeta}$ (after again rescaling ξ in the lower bound (7) on $|I|$, similar as discussed above Theorem 16). □

4 Pseudo-randomness: independent sets in H-free graphs

seudorandom2

The goal of this section is to prove Theorem 7, which concern the construction of pseudo-random subgraphs of a host graph F. As discussed in the introduction, the basic idea is to apply the semi-random independent set algorithm to a suitable e_H-uniform auxiliary hypergraph $\mathcal{H} = (V, E)$, whose vertex-set $V = E_F$ equals the edge-set of the n-vertex graph F and whose edge-set $E = \{E_{H'} : H' \text{ is copy of } H \text{ in } F\}$ encodes the copies of H in F. As we shall discuss, using our main result Theorem 2, we can find a large independent set I in $\mathcal{H}$, which by setting $G := (V_F, I)$ naturally corresponds to an H-free subgraph $G \subseteq F$ on the same vertex-set. Our remaining main task thus reduces to showing that the edges of sufficiently large vertex-subsets $W \subseteq V_G = V_F$ of G are pseudo-random, i.e., that the edge-estimate (11) holds for all vertex-subsets $W \subseteq V_F$ with $|W| = \lceil L(\log n)^{1-1/(e_H-1)} n^{(v_H-2)/(e_H-1)}\rceil$ and $e_{F[W]} \geq \gamma\binom{|W|}{2}$.

To allow for slightly larger generality beyond tracking the number of all edges inside vertex-subsets, for suitable constants $B, T > 0$ we shall in fact keep track of the number of edges in *configurations*

$$\Sigma_s \subseteq \Big\{J \subseteq \tbinom{[n]}{2}\Big\}, \tag{103}$$

def:SigmaS

which are edge-subsets of the complete graph K_n satisfying

$$|\Sigma_s| \leq n^{Ts} \qquad \text{and} \qquad \max_{J \in \Sigma_s} |V(J)| \leq Ts, \tag{104}$$

def:SigmaS:b

where $V(J) := \bigcup_{e \in J} e$ contains all vertices that are in the edges of J, and

$$s := B(\log n)^{1-\frac{1}{r-1}} D^{\frac{1}{r-1}}, \tag{105}$$

eq:def:s

where we defer the choice of the constants $B, T > 0$. Heuristically speaking, in our upcoming arguments the two technical conditions in (104) will ensure that configurations have certain key features of s-element subsets: that the number $|\Sigma_s|$ of configurations is not much larger than the number of s-element subsets, and that each configuration $J \in \Sigma_s$ contains at most $O(s)$ vertices. We now illustrate the versatility of configurations via two concrete examples. First, if our goal was to track the number of edges in every $\lceil s\rceil$-vertex-subset, then we would define Σ_s as the set of all $\binom{W}{2}$ for some $W \subseteq [n]$ of size $|W| = \lceil s\rceil$, which readily satisfies $|\Sigma_s| = \binom{n}{\lceil s\rceil} \leq n^{2s}$ and $\max_{J\in\Sigma_s} V(J) = \lceil s\rceil \leq 2s$, say. Second, if our goal was to track the number of edges in every bipartite subgraph $K_{\lceil s\rceil,\lceil s\rceil}$, then we would define Σ_s as the set of all $U \times W$ for some disjoint $U \subseteq [n]$ and $W \subseteq [n] \setminus U$ of size $|U| = |W| = \lceil s\rceil$, which readily satisfies $|\Sigma_s| \leq \binom{n}{\lceil s\rceil}^2 \leq n^{3s}$ and $\max_{J\in\Sigma_s} V(J) = 2\lceil s\rceil \leq 3s$, say.

In the remainder of this section we fix a graph $H \in \mathfrak{F}$ (see (10)), and focus on the behavior of the semi-random algorithm on the r-uniform hypergraph $\mathcal{H}_{H,F}$ with $r = e_H$ for some n-vertex graph F. For each edge-set $J \in \Sigma_s$ (that satisfy some technical conditions to be discussed), our goal will be to control the number of edges $|J \cap I_i|$ added by the semi-random algorithm, which in turn requires control over the number of edges $|J \cap A_i|$ that can still be added.

4.1 Smooth independence

ments:smooth

The concentration arguments for $|J \cap I_i|$ and $|J \cap A_i|$ will requires some extra wrinkles, since the addition of certain edges can result in fairly large changes in these variables. Similar technical issues also arise in the analysis of the H-free process due to Bohman and Keevash [8]. In attempt to simplify this part of the argument and clarify the relation to previous work, we thus have set up our random variables in such a way that we can reuse (and extend) some existing estimates from the H-free process analysis [8, Section 11] in the semi-random setting of the present paper.

Remark 38 (Polynomial parameter f)**.** *In contrast to Sections 2–3, in this section we shall henceforth always assume that there is a constant $\eta > 0$, such that the parameter*

$$f := n^{\eta} \tag{106}$$

def:f:neta

satisfies assumptions (3)–(5) *and assumption* $D^{\frac{1}{r-1}} \geq f^{\beta}$ *in* (6).

Turning to the details, we shall use the following auxiliary variable $\tilde{Y}_v(i)$ which intuitively counts the number of 'relaxed' neighbors of v: indeed, it can be viewed as relaxation of $Y_v(i)$ defined in Section 2.2, where we remove the requirements of certain vertices in A_i. More precisely, for $v \in A_0$ we define

$$\tilde{Y}_v(i) := \{u \in A_0 \setminus \{v\} : \text{ there exists } e \in \mathcal{H}_H \text{ such that } u, v \in e \text{ and } e \setminus \{u, v\} \subseteq V_i\}. \tag{107}$$

eq:def:tcYv

This enables to us to introduce the following two sets, which contain elements in A_0 and V_i, respectively, that have many relaxed neighbors in J. More precisely, for $J \subseteq \binom{[n]}{2}$, we define

$$B_J(i) := \{v \in A_0 : |\tilde{Y}_v(i) \cap J \cap A_0| \geq f^{-\tau} D^{\frac{1}{r-1}}\},$$
$$P_J(i) := \{v \in V_i : |\tilde{Y}_v(i) \cap J \cap A_0| \geq f^{-\tau} D^{\frac{1}{r-1}}\}.$$

Intuitively, the plan is to indirectly control $|J \cap I_i|$, via an approximation that ignores the effect of elements with big changes. For $J \subseteq \binom{W}{2}$, the following definition intuitively formalizes the idea that (a) this 'ignoring approach' will not change the typical behavior due to smallness of $B_{\binom{W}{2}}(i)$, and (b) that overall not too many elements are ignored due to smallness of $P_{\binom{W}{2}}(i)$. As we shall see, this intuitively means that the resulting approximation can indeed be used to control $|J \cap I_i|$, as desired.

def:smooth

Definition 1. *A graph H has the smooth independence property, if there exist $\tau, \xi_1 > 0$ (that may depend on H, η) such that, for all $w = \Theta\Big((\log n)^{1-\frac{1}{r-1}} D^{\frac{1}{r-1}}\Big)$, the following holds for any hypergraph $\mathcal{H} \subseteq \mathcal{H}_H$. For all $0 < \xi \leq \xi_1$ satisfying* (38) *we have*

$$\mathbb{P}(\mathcal{S}_i \textit{ for all } 0 \leq i \leq m) \geq 1 - n^{-\omega(1)}, \tag{108}$$

eq:pr:smooth

where $\mathcal{S}_i$ denotes the event that, for all $W \subseteq [n]$ of size $|W| = w$, we have

$$|B_{\binom{W}{2}}(i)| \leq f^{-2\tau} D^{\frac{2}{r-1}}, \tag{109}$$

eq:def:BW2i

$$|P_{\binom{W}{2}}(i)| \leq f^{-\tau} D^{\frac{1}{r-1}}. \tag{110}$$

eq:def:PW2i

There are a wide range of graphs having the smooth independent property, including all graphs $H \in \mathfrak{F}$, where the set $\mathfrak{F}$ is defined as in (10). The proof of Lemma 39 reuses (and extends) some estimates from the H-free process analysis [8], and is deferred to Section 4.5. Recall that $\mathfrak{F}$ contains complete graphs K_ℓ with $\ell \geq 4$, cycles C_ℓ with $\ell \geq 4$, complete bipartite graphs $K_{a,a}$ with $a \geq 4$, and hypercubes Q^k with odd $k \geq 5$.

lem:smooth

Lemma 39 (Smooth independence)**.** *Any graph $H \in \mathfrak{F}$ has the smooth independence property.*

4.2 Pseudo-random properties and statement of main theorem

ndstatements

In Sections 4.2–4.3, we henceforth fix a graph H that has the smooth independence property, and set

$$w := Ts = TB(\log n)^{1-\frac{1}{r-1}} D^{\frac{1}{r-1}}, \tag{111}$$

eq:defoffixe

where we defer the choice of the constants $B, T > 0$ (as before). Note that whenever the 'smooth independence' event $\mathcal{S}_i$ from Definition 1 holds, then for all $J \in \Sigma_s$ we have

$$|B_J(i)| \le f^{-2\tau} D^{\frac{2}{r-1}} \quad \text{and} \quad |P_J(i)| \le f^{-\tau} D^{\frac{1}{r-1}}, \tag{112}$$

eq:smooth_bo

since $J \subseteq \binom{W}{2}$ for a suitable $W \subseteq [n]$ to which (109)–(110) apply.

With a view of Theorem 7, we are interested in edge-sets $J \in \Sigma_s$ that are relatively dense. For this reason we introduce

$$\Sigma_{s,\gamma} := \Big\{ J \in \Sigma_s : |J \cap A_0| \ge \gamma \binom{s}{2} \Big\}, \tag{113}$$

eq:defofsigm

where $\gamma > 0$ is a constant. Using the bounds from (112), we will be able to show that during the semi-random independent set algorithm we have $|J \cap A_i| \approx q_i \cdot |J \cap A_0|$ for all steps $0 \le i < m$ and configurations $J \in \Sigma_{s,\gamma}$, which coincides with our pseudo-random intuition from Section 2.3 that yielded $|A_i| \approx q_i |A_0|$ in Section 3.1.2. Using these estimates on $|J \cap A_i|$ we will then be able to control $|J \cap I_m|$ similarly to Section 3.2, where we used estimates for $|A_i|$ to control $|I_m|$. One main conceptual difference to earlier sections is that here we need to establish control for all configurations $J \in \Sigma_{s,\gamma}$, which in our arguments requires us to obtain error probabilities that are small enough to take a union bound over all of the $|\Sigma_{s,\gamma}| \le n^w$ many configurations.

Inspired by Section 2.5, we now define several events that capture pseudo-random properties of the semi-random algorithm. Recalling $\mathfrak{X}_i$ defined as (32), we define the 'good' events

$$\mathfrak{X}_i^+ := \mathfrak{X}_i \cap \tilde{\mathcal{Y}}_i \cap \mathcal{A}_i^+ \cap \mathcal{S}_i \qquad \text{and} \qquad \mathfrak{X}_{\le i}^+ := \cap_{0 \le j \le i} \mathfrak{X}_j^+, \tag{114}$$

cGiPlus

where

$$\tilde{\mathcal{Y}}_i := \Big\{ |\tilde{Y}_v(i)| \le D^{\frac{1}{r-1}} Q^{r-1} \text{ for all } v \in A_0 \Big\}, \tag{115}$$

eq:def:teven

$$\mathcal{A}_i^+ := \Big\{ \tau_i q_i |J \cap A_0| \le |J \cap A_i| \le q_i |J \cap A_0| \text{ for all } J \in \Sigma_{s,\gamma} \Big\}, \tag{116}$$

eq:def:event

$$\mathcal{I}_m^+ := \Big\{ |J \cap I_m| \in [(1-\delta)\mu_J, (1+\delta)\mu_J] \text{ for all } J \in \Sigma_{s,\gamma} \Big\}, \tag{117}$$

eq:def:Isett

with

$$\mu_J = \xi \Big(\frac{\log f}{D} \Big)^{\frac{1}{r-1}} |J \cap A_0|. \tag{118}$$

eq:cmuJ

Note that $\tilde{\mathcal{Y}}_i$ is a variant of $\mathcal{Y}_i$ defined in (34) that concerns relaxed neighbors. Furthermore, $\mathcal{A}_i^+$ and $\mathcal{I}_m^+$ are natural variants of $\mathcal{A}_i$ and $\mathcal{I}_m$ defined in (33) and (31) that apply to $|J \cap A_i|$ and $|J \cap I_m|$, respectively.

Remark 40. *The event $\mathfrak{X}_0^+$ holds deterministically.*

We are now ready to state our main technical result regarding independent sets, which states that the pseudo-random events $\mathcal{I}_m^+$ and $\mathfrak{X}_{\le m}^+$ both hold with very high probability. For a graph H, recall that $\mathcal{H}_H$ has vertex-set $E(K_n)$ and edge-set the collection of the edge-sets of all H-copies in K_n. As we shall see below, Theorem 41 combined with the smooth independence result Lemma 39 imply Theorem 7 (see Section 4.4 for a proof of Corollary 8).

:boundofIinW

Theorem 41. *Fix constants $\delta, \gamma, \eta, \beta \in (0,1]$, and a graph H that has the smooth independence property. Assume that the subhypergraph $\mathcal{H} \subseteq \mathcal{H}_H(n)$ has $|V_{\mathcal{H}}| = N$ vertices. Setting the parameter $f := n^\eta$ as in (106), assume that $\mathcal{H}$ satisfies assumptions (3)–(5) and assumption $D^{\frac{1}{r-1}} \ge f^\beta$ in (6). Let $\xi_1 > 0$ be as in the definition of the smooth independence property (see Definition 1). Then, for any $\xi, T, B > 0$ satisfying*

$$\xi \le \min\Big\{ \Big(\frac{\min\{\tau, \beta\}}{37r} \Big)^{\frac{1}{r-1}}, \xi_1 \Big\} \qquad \text{and} \qquad B \ge \frac{288T}{\delta^2 \xi \gamma \eta^{\frac{1}{r-1}}}, \tag{119}$$

eq:technical

we have

$$\mathbb{P}(\mathcal{I}_m^+ \cap \mathfrak{X}_{\le m}^+) \ge 1 - n^{-\omega(1)}.$$

Remark 42. *In Theorem 41, since $N \le \binom{n}{2} \le n^2$ and $f = n^\eta$, for any constant $c > 0$ it immediately follows that the assumption $f \ge (\log N)^c$ in* (6) *holds for all sufficiently large n.*

The proof strategy for Theorem 41 is similar to that of Theorem 16, except that the argument for the event $\mathcal{A}_i^+$ we will rely on the smooth independence property to make the union bound argument work (by limiting the effects of certain bad contributions that would otherwise spoil the probability estimates). This is also the core reason why we require $f = n^\eta$ in Theorem 41, rather than the logarithmic bound in Theorem 2. The proof of Theorem 41 is given in Section 4.3.

Proof of Theorem 7 based on Theorem 41. Since any $H \in \mathfrak{F}$ is strictly 2-balanced with minimum degree at least two, by Lemma 4, the hypergraph $\mathcal{H} = \mathcal{H}_{H,F}$ (recalling that its vertex-set is E_F and edge-set is $\{E_{H'} \mid H' \text{ is an } H\text{-copy in } F\}$), which is a subgraph of $\mathcal{H}_H$, satisfies the assumption (3)–(5) and assumption $D^{\frac{1}{r-1}} \ge f^\beta$ in (6) for $D = \lambda n^{v_H - 2}$ and $f = n^\eta$ for some constants $\lambda, \eta, \beta > 0$. Furthermore, by Lemma 39, $H \in \mathfrak{F}$ has the smooth independence property, so we henceforth fix τ as in Definition 1. At this point, we have verified the assumptions of Theorem 41.

Note that by the above choice of D and f we have

$$\log f = \eta \log n \quad \text{and} \quad D^{\frac{1}{r-1}} = \lambda^{1/(e_H-1)} n^{(v_H-2)/(e_H-1)}.$$

Therefore to prove Theorem 7, for any $\delta, \gamma \in (0,1]$, taking $T = 2$, there exist ξ and B satisfying the assumption of Theorem 41. In view of (105), we can set $L = B\lambda^{1/(e_H-1)}$, and in view of (118), we can set $\xi_0 := \xi(\eta/\lambda)^{1/(e_H-1)}$. Recalling the discussion about the number of configurations satisfying $|\Sigma_s| \le n^{2s} = n^{Ts} = n^w$ at the beginning of this section, it completes the proof of Theorem 7. □

We now specify the choices of the constants ξ used in the proof of Theorem 41. Since we are reusing the event $\mathfrak{X}_i$ from Section 2.5, here our choices need to be compatible with the proof of Theorem 16. Recall that around (38) in Section 2.6 we discussed that the proof of Theorem 16 works for any $\xi \in (0, \infty)$ satisfying $\xi \le (\beta/(37r))^{\frac{1}{r-1}}$. Taking τ into consideration, for concreteness we thus here set

$$\xi := \Big(\frac{\min\{\tau, \beta\}}{37r}\Big)^{\frac{1}{r-1}}, \tag{120}$$ eq:defofxi2

and then define $c = c(\xi)$ as in (38). These choices lead to the following convenient technical estimates.

xicstaucbeta **Remark 43.** *Assume that the assumptions of Theorem 41 hold. By arguing similar as for* (39) *and* (40) *in Lemma 18, by taking the stronger assumption $f = n^\eta$ into account, there exists $\eta_0 > 0$ such that the following estimates are valid for all $0 \le i \le m$:*

$$\max_{\substack{x \ge 1/2 = \rho, \\ 0 \le y \le 2r}} \sigma^x \pi_i^y = o\big((\log f)^{-2}\big), \tag{121}$$ eq:sigmapine

$$\frac{\max_{-6r \le j,k,x,y,z \le 6r} \sigma^x q_i^j \pi_i^y Q^k (\log f)^z}{\min\{f,\ f^\beta,\ f^\tau,\ D^{\frac{1}{r-1}}\}} \ll n^{-\eta_0}. \tag{122}$$ eq:assumptio

4.3 Proof of Theorem 41

fofTheorem39

In this section, we prove the nibble result Theorem 41 by establishing the following auxiliary lemma.

a:nibbleforJ **Lemma 44.** *Under the assumption of Theorem 41, we have*

$$\max_{0 \le i < m} \mathbb{P}(\neg \mathcal{A}_{i+1}^+ \cap \mathcal{N}_{i+1} \cap \tilde{\mathfrak{Y}}_{i+1} \cap \mathcal{S}_{i+1} \mid \mathfrak{X}_{\le i}^+) \le n^{-\omega(1)}, \tag{123}$$ eq:Avi+1^+

$$\mathbb{P}(\neg \tilde{\mathfrak{Y}}_i \text{ for some } 0 \le i \le m) \le n^{-\omega(1)}, \tag{124}$$ eq:negevento

$$\mathbb{P}(\neg \mathfrak{X}_i \text{ for some } 0 \le i \le m) \le n^{-\omega(1)}. \tag{125}$$ eq:negcG

$$\mathbb{P}(\neg \mathcal{I}_m^+ \cap \mathfrak{X}_{\le m}^+) \le n^{-\omega(1)}, \tag{126}$$ eq:Isettm

Proof of Theorem 41 (assuming Lemma 44). The argument is very similar to the proof of Theorem 16 in Section 3 (using Lemma 44 instead of Lemma 25). Turning to the details, recall that $m \le n^{O(1)}$, and that $\mathfrak{X}_0$ holds deterministically by Remark 17. In view of the good events $\mathfrak{X}_i^+ = \mathfrak{X}_i \cap \tilde{\mathcal{Y}}_i \cap \mathcal{A}_i^+ \cap \mathcal{S}_i$ and $\mathfrak{X}_i = \mathcal{A}_i \cap \mathcal{Y}_i \cap \mathcal{N}_i \cap \mathcal{P}_i$, by combining inequalities (123)–(125) with the assumed bound (108) for the event that $\neg\mathcal{S}_i$ for some $0 \le i \le m$, we readily obtain that

$$\begin{aligned} \mathbb{P}(\neg\mathfrak{X}^+_{\le m}) &\le 3 \cdot n^{-\omega(1)} + \sum_{0\le i<m} \mathbb{P}\big(\neg\mathfrak{X}^+_{i+1} \cap \mathfrak{X}_{i+1} \cap \tilde{\mathcal{Y}}_{i+1} \cap \mathcal{S}_{i+1} \cap \mathfrak{X}^+_{\le i}\big) \\ &\le 3 \cdot n^{-\omega(1)} + \sum_{0\le i<m} \mathbb{P}(\neg\mathcal{A}^+_{i+1} \cap \mathcal{N}_{i+1} \cap \tilde{\mathcal{Y}}_{i+1} \cap \mathcal{S}_{i+1} \mid \mathfrak{X}^+_{\le i}) \\ &\le O(m) \cdot n^{-\omega(1)} \le n^{-\omega(1)}, \end{aligned} \tag{127}$$

which together with inequality (126) completes the proof of Theorem 41. □

As recorded in Remark 26, the arguments in Section 3 already establish inequality (125), so to prove Lemma 44 it remains to prove the three inequalities (123)–(124) and (126) in the following subsections.

4.3.1 Event $\mathcal{A}^+_{i+1}$: available vertices in configurations

sub:AiinJ

In this subsection we prove estimate (123) via Theorem 45 below, which concerns concentration of the number $|J \cap A_{i+1}|$ of available vertices in all configurations $J \in \Sigma_{s,\gamma}$ (i.e., which can potentially be added in future steps).

einlargesets **Theorem 45.** *We have* $\mathbb{P}(\neg\mathcal{A}^+_{i+1} \cap \mathcal{N}_{i+1} \cap \tilde{\mathcal{Y}}_{i+1} \cap \mathcal{S}_{i+1} \mid \mathcal{F}_i) \le n^{-\omega(1)}$ *when* $\mathfrak{X}^+_{\le i}$ *holds.*

The following proof of Theorem 45 is very similar to the proof of Theorem 32 in Section 3.1.2, which simply concerned the number $|A_{i+1}|$ of available vertices. As we shall see, here the assumed extra smoothness property will be crucial, since it enables a union bound argument over all configurations $J \in \Sigma_{s,\gamma}$, of which there are $|\Sigma_{s,\gamma}| \le n^w$ many.

Proof. For a given $J \in \Sigma_{s,\gamma}$, our strategy for proving a concentration bound for $|J \cap A_{i+1}|$ as in (116) proceeds in two steps. We first enlarge $|J \cap A_{i+1}|$ a little bit by introducing

$$X_J(i+1) := \Big|\big\{v \in J \cap A_i : v \in A_{i+1} \text{ or there exists } u \in B_J(i) \cap \Gamma_{i+1} \text{ such that } v \in \tilde{Y}_u(i)\big\}\Big|, \tag{128}$$ eq:XJi1

and then show that $|J \cap A_{i+1}|$ is close to $|X_J(i+1)|$. To avoid clutter, we shall omit the condition on $\mathcal{F}_i$ from our notations as we did in Section 3.

We start with an upper bound on $X_J(i+1)$. Using $A_{i+1} \subseteq A_i$ and $\Gamma_{i+1} \subseteq A_i$, we have

$$X_J(i+1) \le \sum_{v \in J \cap A_i} \mathbb{1}_{\{v \notin C_{i+1}\}} + \sum_{u \in B_J(i) \cap A_i} \mathbb{1}_{\{u \in \Gamma_{i+1}\}} |\tilde{Y}_u(i) \cap J \cap A_i|. \tag{129}$$ eq:trade-off

Note that the smooth independence property $\mathfrak{X}^+_{\le i} \subseteq \mathcal{S}_i$ implies $|B_J(i)| \le f^{-2\tau} D^{\frac{2}{r-1}}$ (see Definition 1). Hence using $\mathfrak{X}^+_{\le i} \subseteq \mathcal{A}^+_i \cap \tilde{\mathcal{Y}}_i \cap \mathcal{S}_i$ together with Lemma 27, the inequalities $|J \cap A_0| \gg D^{\frac{2}{r-1}}$ and $D \le n^C$, and the estimate inequality (122) of Remark 43, it follows that

$$\begin{aligned} \mathbb{E}X_J(i+1) &\le q_i |J \cap A_0| \Big(\frac{q_{i+1}}{q_i} - \frac{1}{2}(r-1)\sigma^{1+\rho}\pi_i^{\max\{r-3,0\}}\Big) + f^{-2\tau} D^{\frac{2}{r-1}} \cdot D^{\frac{1}{r-1}} Q^{r-1} \cdot \frac{\sigma}{q_i D^{\frac{1}{r-1}}} \\ &\le q_{i+1}|J \cap A_0| - \frac{1}{4}(r-1)\sigma^{1+\rho} q_i \pi_i^{\max\{r-3,0\}} |J \cap A_0|. \end{aligned}$$

For the upper bound on $X_J(i+1)$, we shall apply bounded-difference inequality (see (43) in Remark 21) with

$$t := \frac{1}{4}(r-1)\sigma^{1+\rho} q_i \pi_i^{\max\{r-3,0\}} |J \cap A_0|.$$

To this end we need to bound the parameter C from Remark 21 and the parameter λ from (57) associated with the random variable $X_J(i+1)$ defined in (128). Let us first focus on the vertex-effect c_u (an upper bound on how much $X_J(i+1)$ changes if we alter whether or not u is in Γ_{i+1}) for a vertex $u \in A_i$. Note the trade-off between the first and second summation on the right-hand side of (129): if $u \in B_J(i) \cap A_i$ is included into Γ_{i+1}, then for any $v \in \tilde{Y}_u(i) \cap J \cap A_i = Y_u(i) \cap J \cap A_i$, the vertex v becomes 'closed' so is not counted by the first summation, but v is counted by the second one; vice versa for the case that $u \in B_J(i) \cap A_i$ and is not included into Γ_{i+1}. Therefore for $u \in B_J(i) \cap A_i$ altering whether or not u is in Γ_{i+1} does not change the value of $X_J(i+1)$, i.e., $c_u = 0$. For $u \in A_i \setminus B_J(i)$, $c_u \leq |Y_u(i) \cap J \cap A_i| \leq |\tilde{Y}_u(i) \cap J \cap A_0| \leq f^{-\tau} D^{\frac{1}{r-1}}$. Putting both cases together, we infer that for any vertex $u \in A_i$, the vertex-effect c_u is at most

$$c_u \leq |\tilde{Y}_u(i) \cap J \cap A_0| \leq f^{-\tau} D^{\frac{1}{r-1}}. \tag{130}$$

eq:cu:bound:

As $\mathfrak{X}^+_{\leq i} \subseteq \tilde{\mathcal{Y}}_i$, by mimicking the proof of Lemma 30 (using that $z \in \tilde{Y}_u(i)$ implies $u \in \tilde{Y}_z(i)$) it follows that

$$\begin{aligned}\sum_{u \in A_i} c_u \leq \sum_{u \in A_i} |\tilde{Y}_u(i) \cap J \cap A_0| &= \sum_{z \in J \cap A_0} \sum_{u \in A_i} \mathbb{1}_{\{z \in \tilde{Y}_u(i)\}} \\ \leq \sum_{z \in J \cap A_0} \sum_{u \in A_i} \mathbb{1}_{\{u \in \tilde{Y}_z(i)\}} &\leq \sum_{z \in J \cap A_0} |\tilde{Y}_z(i)| \leq D^{\frac{1}{r-1}} Q^{r-1} |J \cap A_0|.\end{aligned}$$

Furthermore, the stabilization-effect satisfies $\hat{c}_u \leq \mathbb{1}_{\{u \in J \cap A_i\}}$. Putting the above-discussed estimates together, we infer that the parameter λ from (57) satisfies

$$\begin{aligned}\lambda &\leq p_i \max_{u \in A_i} c_u \cdot \sum_{u \in A_i} c_u + \sum_{u \in A_i} \mathbb{1}_{\{u \in J \cap A_i\}} \\ &\leq \frac{\sigma}{q_i D^{\frac{1}{r-1}}} f^{-\tau} D^{\frac{1}{r-1}} \cdot D^{\frac{1}{r-1}} Q^{r-1} |J \cap A_0| + q_i |J \cap A_0|,\end{aligned}$$

and that the parameter C from Remark 21 satisfies

$$C = \max\Big\{ \max_{u \in A_i} c_u \, , \, \max_{u \in A_i} \hat{c}_u \Big\} \leq \max\big\{ f^{-\tau} D^{\frac{1}{r-1}}, \, 1 \big\}.$$

Recall that $J \in \Sigma_{s,\gamma}$ satisfies $|J \cap A_0| \geq \gamma \binom{s}{2}$ by (113). Invoking the bounded-difference inequality (43), using the estimate (122) of Remark 43 it follows that, say,

$$\mathbb{P}\Big(X_J(i+1) \geq q_{i+1} |J \cap A_0| \Big) \leq \exp\left(- \frac{t^2}{2(\lambda + Ct)} \right) \leq \exp\big(-(\log n)^2 w \big). \tag{131}$$

eq:upperboun

Regarding the lower bound for $|X_J(i+1)|$, we say that a vertex $v \in J \cap A_i$ is closed by a vertex $u \in B_J(i)$ if $u \in \Gamma_{i+1}$ and $v \in Y_u(i) = \tilde{Y}_u(i) \cap A_i$. Note that any such vertex v is still counted by $X_J(i+1)$. Therefore

$$X_J(i+1) \geq \underbrace{\sum_{v \in J \cap A_i} \mathbb{1}_{\{v \notin C_{i+1} \text{ or } v \text{ is closed by some vertex in } B_J(i)\}}}_{=:Z_1} - \underbrace{\sum_{v \in J \cap A_i} \mathbb{1}_{\{v \in \Gamma_{i+1}\}}}_{=:Z_0} - \sum_{j=2}^{r-1} \underbrace{\sum_{v \in J \cap A_i} \mathbb{1}_{\{v \in C^{(j)}_{i+1}\}}}_{=:Z_j}. \tag{132}$$

eq:decompose

Then to prove a lower bound on $X_J(i+1)$, we shall prove a lower bound on Z_1 and upper bounds on Z_0 and Z_j for $2 \leq j \leq r-1$. To be specific about the targeting bounds, setting the error term $t := \sigma^{1+\rho} q_{i+1} \pi_i^{\max\{r-3,0\}} |J \cap A_0|$, we shall prove that for a given $J \in \Sigma_{s,\gamma}$,

$$\mathbb{P}\Big(Z_1 \leq \tau_i q_{i+1} |J \cap A_0| - 2rt \Big) \leq \exp\big(-(\log n)^2 w \big). \tag{133}$$

eq:boundZ1

$$\mathbb{P}(Z_0 \geq t) \leq \exp\big(-(\log n)^2 w \big), \tag{134}$$

eq:boundZ0

and for all $J \in \Sigma_{s,\gamma}$

$$\mathbb{P}(Z_j \geq t \text{ and } \mathcal{N}_{i+1} \cap \tilde{\mathcal{Y}}_{i+1} \cap \mathcal{S}_{i+1}) \leq n^{-\omega(1)}. \tag{135}$$

eq:boundZj

Note that the difference in the probabilities in (133)–(135) is just because we shall use the smooth independence property $\mathcal{S}_{i+1}$ to prove (135).

In order to bound Z_1 from below as in (133), since $\mathfrak{X}_{\leq i} \subseteq \mathcal{A}_i^+$ implies $|J \cap A_i| \geq \tau_i q_i |J \cap A_0|$, using Lemma 27 together with $\tau_i \leq 1$ and $\frac{q_{i+1}}{q_i} \sim 1$ (by Remark 28) we obtain that

$$\begin{aligned} \mathbb{E}Z_1 &\geq \tau_i q_i |J \cap A_0| \Big(\frac{q_{i+1}}{q_i} - \frac{3}{2}(r-1)\sigma^{1+\rho}\pi_i^{\max\{r-3,0\}} \Big) \\ &\geq \tau_i q_{i+1} |J \cap A_0| - \underbrace{2(r-1)\sigma^{1+\rho} q_{i+1} \pi_i^{\max\{r-3,0\}} |J \cap A_0|}_{=2(r-1)t}. \end{aligned} \tag{136}$$

eq:EZ1:def:t

The plan is to invoke the bounded-difference inequality (Remark 21). Note that by definition of Z_1, for a vertex $u \in A_i$, using the same trade-off argument as for (130) we again infer that the vertex-effect satisfies $c_u \leq |\tilde{Y}_u(i) \cap J \cap A_0| \leq f^{-\tau} D^{\frac{1}{r-1}}$. Using Lemma 30 and $\mathfrak{X}_i^+ \subseteq \mathcal{A}_i^+ \cap \mathcal{Y}_i$ we thus infer that

$$\sum_{u \in A_i} c_u \leq \sum_{u \in A_i} |Y_u(i) \cap J \cap A_i| \leq D^{\frac{1}{r-1}} (r-1) q_i^2 \pi_i^{r-2} |J \cap A_0|,$$

and that the stabilization-effect satisfies $\hat{c}_u \leq \mathbb{1}_{\{u \in J \cap A_i\}}$. Putting these together, we note that parameters λ and C from (57) and Remark 21 satisfy

$$\lambda \leq p_i \cdot f^{-\tau} D^{\frac{1}{r-1}} \cdot D^{\frac{1}{r-1}} (r-1) q_i^2 \pi_i^{r-2} |J \cap A_0| + q_i |J \cap A_0|$$

as well as

$$C = \max\Big\{ \max_{u \in A_i} c_u \ , \ \max_{u \in A_i} \hat{c}_u \Big\} \leq \max\{ f^{-\tau} D^{\frac{1}{r-1}}, \ 1 \}.$$

Invoking the bounded-difference inequality (43) with t as defined in (136), using $J \in \Sigma_{s,\gamma}$ together with the technical assumption (122) of Remark 43 it follows that

$$\begin{aligned} &\mathbb{P}\Big(Z_1 \leq \tau_i q_{i+1} |J \cap A_0| - 2r\sigma^{1+\rho} q_{i+1} \pi_i^{\max\{r-3,0\}} |J \cap A_0| \Big) \\ &\qquad \leq \exp\left(- \frac{t^2}{2(\lambda + Ct)} \right) \leq \exp\big(- (\log n)^2 w \big), \end{aligned} \tag{137}$$

which proves (133).

We now derive the upper bound for Z_0 in (134). Since $\mathfrak{X}_{\leq i}^+ \subseteq \mathcal{A}_i^+$ implies $|J \cap A_i| \leq q_i |J \cap A_0|$, note that using $J \in \Sigma_{s,\gamma}$ and the technical estimate (122) of Remark 43, we infer that

$$\mathbb{E}Z_0 \leq q_i |J \cap A_0| \frac{\sigma}{q_i D^{\frac{1}{r-1}}} \ll \sigma^{1+\rho} q_{i+1} \pi_i^{\max\{r-3,0\}} |J \cap A_0| =: \mu,$$

with $\mu = \Omega\big((\log n)^2 w \big)$, say. Invoking a standard Chernoff bound (see Remark 22), it follows that

$$\begin{aligned} \mathbb{P}\Big(Z_0 \geq \sigma^{1+\rho} q_{i+1} \pi_i^{\max\{r-3,0\}} |J \cap A_0| \Big) &\leq \mathbb{P}\Big(Z_0 \geq \mathbb{E}Z_0 + \mu/2 \Big) \\ &\leq \exp\Big(-\Omega(\mu) \Big) \leq \exp\Big(- \Omega\Big((\log n)^2 w \Big) \Big), \end{aligned} \tag{138}$$

boundofZ0

which proves (134).

Next we turn to the proof of (135). For each $2 \leq j \leq r-1$, to prove an upper bound for Z_j as defined in (132), we first decompose it into two parts. Indeed, note that

$$Z_j = \Big| C_{i+1}^{(j)} \cap J \cap A_i \Big| \leq \Big| \hat{C}_{i+1,J}^{(j)} \Big| + \Big| \Big(C_{i+1}^{(j)} \setminus \hat{C}_{i+1,J}^{(j)} \Big) \cap \Big(J \cap A_i \Big) \Big|, \tag{139}$$

twopartofZj

where we (with foresight) used

$$\begin{aligned} \hat{C}_{i+1,J}^{(j)} := \Big\{ v \in J \cap A_i : & \text{ there exists } u \in P_J(i+1) \cap \Gamma_{i+1} \text{ with } u \neq v \text{ and } e \in E_{\mathcal{H}} \text{ s.t.} \\ & u, v \in e, \ |(e \setminus \{u,v\}) \cap \Gamma_{i+1}| = j-1, \ |(e \setminus \{u,v\}) \cap V_i| = r-j-1 \Big\}. \end{aligned}$$

Note that for any vertex $v \in J \cap A_i$ in the above definition, there exists a vertex $u \in P_J(i+1)$ such that $v \in \tilde{Y}_u(i+1)$. Exploiting the smooth independence property $\mathcal{S}_{i+1}$ to infer $|P_J(i+1)| \le f^{-\tau} D^{\frac{1}{r-1}}$, and the relaxed neighbor event $\tilde{\mathcal{Y}}_{i+1}$ to infer $|\tilde{Y}_u(i+1)| \le D^{\frac{1}{r-1}} Q^{r-1}$, for all $J \in \Sigma_{s,\gamma}$ it thus follows that

$$\left|\hat{C}^{(j)}_{i+1,J}\right| \le \sum_{u \in P_J(i+1)} |\tilde{Y}_u(i+1)| \le f^{-\tau} D^{\frac{1}{r-1}} \cdot D^{\frac{1}{r-1}} Q^{r-1}. \tag{140}$$

firstpartofZ

For the second term in (139), note that for any vertex $v \in \left(C^{(j)}_{i+1} \setminus \hat{C}^{(j)}_{i+1}\right) \cap \left(J \cap A_i\right)$, there exists some vertex-set $U \in Y_{v,j}(i)$ such that $U \cap P_J(i+1) = \varnothing$ and $U \subseteq \Gamma_{i+1}$. Therefore

$$\left|\left(C^{(j)}_{i+1} \setminus \hat{C}^{(j)}_{i+1}\right) \cap \left(J \cap A_i\right)\right| \le \sum_{(U,v): v \in J \cap A_i, U \in Y_{v,j}(i), U \cap P_J(i+1) = \varnothing} \mathbb{1}_{\{U \subseteq \Gamma_{i+1}\}} =: Z_{j,2}.$$

Combined with $\mathfrak{X}^+_{\le i} \subseteq \mathcal{A}^+_i \cap \mathcal{Y}_i$, we infer that

$$\begin{aligned}
\mathbb{E} Z_{j,2} &\le |J \cap A_i| \cdot \max_{v \in A_0} |Y_{v,j}(i)| \cdot p_i^j \\
&\le q_i |J \cap A_0| \cdot \binom{r-1}{j} q_i^j \pi_i^{r-1-j} D^{\frac{j}{r-1}} \cdot \Big(\frac{\sigma}{q_i D^{\frac{1}{r-1}}}\Big)^j \\
&= \binom{r-1}{j} \sigma^j q_i \pi_i^{r-1-j} |J \cap A_0|.
\end{aligned}$$

We shall invoke Theorem 23 via Remark 24 to bound the upper-tail of $Z_{j,2}$. Using notation from Theorem 23 and Remark 24, we have $\xi(a) := \mathbb{1}_{\{a \in \Gamma_{i+1}\}}$ for vertices $a \in A_i$, and $Z_{j,2} = \sum_{\alpha \in \mathcal{I}} Y_\alpha$ for variables $Y_\alpha := \mathbb{1}_{\{U \subseteq \Gamma_{i+1}\}}$, where the index set $\mathcal{I}$ consists of all ordered pairs $\alpha = (U, v)$ that satisfy $v \in J \cap A_i$, $U \in Y_{v,j}(i)$, and $U \cap P_J(i+1) = \varnothing$. Next we shall bound the parameter C from Remark 24 associate with $Z_{j,2}$. To this end, given any $(U, v) \in \mathcal{I}$ with $U \subseteq \Gamma_{i+1}$, it suffices to bound the number of relevant $(\hat{U}, \hat{v}) \in \mathcal{I}$, i.e., which satisfy $U \cap \hat{U} \neq \varnothing$ and $\hat{U} \subseteq \Gamma_{i+1}$. For this we first derive one extra constraint. Since we require $\hat{U} \in Y_{\hat{v},j}(i)$, by definition of $Y_{\hat{v},j}(i)$ in (23), there must be an edge $\hat{e}$ satisfying $\hat{v} \in \hat{e}$, $\hat{U} \subseteq \hat{e} \setminus \{\hat{v}\}$, $\hat{e} \setminus (\{\hat{v}\} \cup \hat{U}) \subseteq V_i$. Since we require $\hat{U} \subseteq \Gamma_{i+1} \subseteq V_{i+1}$, then for any $\hat{u} \in \hat{U}$ by definition of $\tilde{Y}_{\hat{u}}(i+1)$, we have $\hat{v} \in \tilde{Y}_{\hat{u}}(i+1) \cap J \cap A_0$, and by $V_{i+1} = V_i \cup \Gamma_{i+1}$,

$$|(\hat{e} \setminus \{\hat{u}, \hat{v}\}) \cap V_{i+1}| = |e \setminus (\{\hat{v}\} \cup \hat{U})| + |\hat{U} \setminus \{\hat{u}\}| = (r-j-1) + (j-1) = r-2. \tag{141}$$

eq:eextracon

With this extra constraint in hand, we are now ready to count the number of relevant $(\hat{U}, \hat{v}) \in \mathcal{I}$, as discussed. As we require $U \cap \hat{U} \neq \varnothing$, there are at most r ways to choose a vertex $\hat{u} \in U$ to be in the intersection. Then there are at most $|\tilde{Y}_{\hat{u}}(i+1) \cap J \cap A_0|$, which by the assumption $\hat{U} \cap P_J(i+1) = \varnothing$ is at most $f^{-\tau} D^{\frac{1}{r-1}}$, ways to choose $\hat{v} \in \tilde{Y}_{\hat{u}}(i+1) \cap J \cap A_0$. Then there are at most $\Delta_{2,r-2}(i+1)$ ways to choose an edge $\hat{e}$ containing $\{\hat{u}, \hat{v}\}$ and satisfying (141). Once $\hat{e}$ is chosen, there are at most 2^r ways to choose $\hat{U}$ contained in $\hat{e}$. Therefore whenever the event $\mathcal{N}_{i+1}$ holds, it follows that the parameter C associated with $Z_{j,2}$ satisfies

$$C \le r \cdot f^{-\tau} D^{\frac{1}{r-1}} \cdot \Delta_{2,r-2}(i+1) \cdot 2^r \le r 2^r f^{-\tau} D^{\frac{1}{r-1}} Q^{r-1}.$$

Invoking Theorem 23 via Remark 24, using (122) from Remark 43 it follows that

$$\begin{aligned}
&\mathbb{P}\Big(Z_{j,2} \ge 2\binom{r-1}{j} \sigma^j q_i \pi_i^{r-1-j} |J \cap A_0| \text{ and } \mathcal{N}_{i+1}\Big) \\
&\qquad \le \exp\Bigg(- \frac{\binom{r-1}{j}^2 \sigma^{2j} q_i^2 \pi_i^{2(r-1-j)} |J \cap A_0|^2}{4r 2^r f^{-\tau} D^{\frac{1}{r-1}} Q^{r-1} \cdot \binom{r-1}{j} \sigma^j q_i \pi_i^{r-1-j} |J \cap A_0|}\Bigg) \le \exp\Big(- (\log n)^2 w\Big).
\end{aligned}$$

Taking a union bound over all $2 \le j \le r-1$ and $J \in \Sigma_{s,\gamma}$, using $|\Sigma_{s,\gamma}| \le n^w$ it readily follows that

$$\sum_{2 \le j \le r-1} \sum_{J \in \Sigma_{s,\gamma}} \mathbb{P}\Big(Z_{j,2} \ge 2\binom{r-1}{j} \sigma^j q_i \pi_i^{r-1-j} |J \cap A_0| \text{ and } \mathcal{N}_{i+1}\Big) \le n^{-\omega(1)}. \tag{142}$$

eq:probbound

By (122) in Remark 43, and by $\frac{q_i}{q_{i+1}} \sim 1$ via Remark 28, $\sigma \le \pi_i$, and $\rho < 1$, for $2 \le j \le r-1$,

$$f^{-\tau} D^{\frac{2}{r-1}} Q^{r-1} \ll 3\binom{r-1}{j} \sigma^j q_i \pi_i^{r-1-j} |J \cap A_0| \ll \sigma^{1+\rho} q_{i+1} \pi_i^{\max\{r-3,0\}} |J \cap A_0|.$$

Combining this with (142) and (140), by (139), we have

$$\mathbb{P}\Big(\{Z_j > \sigma^{1+\rho} q_{i+1} \pi_i^{\max\{r-3,0\}} |J \cap A_0| \text{ for some } 2 \le j \le r-1 \text{ and } J \in \Sigma_{s,\gamma}\} \cap \mathcal{N}_{i+1} \cap \tilde{\mathcal{Y}}_{i+1} \cap \mathcal{S}_{i+1}\Big) \le n^{-\omega(1)}. \tag{143}$$

boundofZj

Thus we finish the proof of (133)–(135), which together with (132) and a union bound over all $J \in \Sigma_{s,\gamma}$ (for (133) and (134)) infers that

$$\mathbb{P}\Big(\{X_J(i+1) \le \tau_i q_{i+1} |J \cap A_0| - 4rt \text{ for some } J \in \Sigma_{s,\gamma}\} \cap \mathcal{N}_{i+1} \cap \tilde{\mathcal{Y}}_{i+1} \cap \mathcal{S}_{i+1}\Big) \ge 1 - n^{-\omega(1)}. \tag{144}$$

eq:lowerboun

We are now ready to prove concentration of $|J \cap A_{i+1}|$. For its upper bound, by (131) and taking a union bound over all $J \in \Sigma_{s,\gamma}$, with probability $1 - n^{-\omega(1)}$ for all $J \in \Sigma_{s,\gamma}$ we have

$$|J \cap A_{i+1}| \le |X_J(i+1)| \le q_{i+1} |J \cap A_0|.$$

For the remaining lower bound on $|J \cap A_{i+1}|$, it suffices to show that the difference between $|J \cap A_{i+1}|$ and $X_J(i+1)$ is small. To see this, obverse that by smooth independence property, (124) in Lemma 44 (to be proved in Section 4.3.3), and (122) in Remark 43, with probability $1 - n^{-\omega(1)}$, for all $J \in \Sigma_{s,\gamma}$ we have

$$\begin{aligned} |X_J(i+1) \setminus (J \cap A_{i+1})| &\le \sum_{u \in P_J(i+1)} |\tilde{Y}_u(i+1)| \le f^{-\tau} D^{\frac{1}{r-1}} \cdot D^{\frac{1}{r-1}} Q^{r-1} \\ &\ll \underbrace{\sigma^{1+\rho} q_{i+1} \pi_i^{\max\{r-3,0\}} |J \cap A_0|}_{=t}. \end{aligned} \tag{145}$$

eq:boundofdi

Since $|J \cap A_{i+1}| = |X_J(i+1)| - |X_J(i+1) \setminus (J \cap A_{i+1})|$, using (144) for the lower bound on the former term and (145) for the upper bound on the latter term, and estimating τ_{i+1} in the same way as the end in the proof of Theorem 32 so that $\tau_i q_{i+1} |J \cap A_0| - 4rt - t \ge \tau_{i+1} q_{i+1} |J \cap A_0|$, we have

$$\begin{aligned} &\mathbb{P}\Big(\big\{|J \cap A_{i+1}| < \tau_{i+1} q_{i+1} |J \cap A_0| \text{ for some } J \in \Sigma_{s,\gamma}\big\} \cap \mathcal{N}_{i+1} \cap \tilde{\mathcal{Y}}_{i+1} \cap \mathcal{S}_{i+1}\Big) \\ \le &\mathbb{P}(\{X_J(i+1) \le \tau_i q_{i+1} |J \cap A_0| - 4rt \text{ for some } J \in \Sigma_{s,\gamma}\} \cap \mathcal{N}_{i+1} \cap \tilde{\mathcal{Y}}_{i+1} \cap \mathcal{S}_{i+1}\Big) \\ &+ \mathbb{P}(|X_J(i+1) \setminus (J \cap A_{i+1})| \ge t \text{ for some } J \in \Sigma_{s,\gamma}) \\ \le &n^{-\omega(1)}, \end{aligned}$$

which completes the proof. □

4.3.2 Event $\mathcal{I}_m^+$: size of $|J \cap I_m|$ for $J \in \Sigma_{s,\gamma}$

sub:IminJ

In this subsection we prove estimate (126) via Theorem 46 below, which concerns concentration of the number of $|J \cap I_m|$ vertices contain in the final independent set I_m that are also in J, for all configuration $J \in \Sigma_{s,\gamma}$.

thm:Isettm **Theorem 46.** *We have* $\mathbb{P}(\neg \mathcal{I}_m^+ \cap \mathfrak{X}_{\le m}^+) \le n^{-\omega(1)}$.

The following based proof of Theorem 46 is very similar to the proof of Theorem 33 in Section 3.2, which simply concerned the number $|I_m|$ vertices in the final independent set I_m. Here the definition of configurations will be crucial, since it enables a union bound argument over all $J \in \Sigma_{s,\gamma}$.

boundofXYinW **Lemma 47.** *Let $\mathcal{T}_J$ denote the event that the following bounds hold:*

$$X_J := \sum_{0 \le i < m} |J \cap A_i \cap \Gamma_{i+1}| \in \Big[(1 - \frac{\delta}{2})\mu_J^-, (1 + \frac{\delta}{2})\mu_J^+\Big], \tag{146}$$

eq:boundXJ

$$Y_J := \sum_{0 \le i < m} |J \cap A_i \cap V(\mathcal{D}_{i+1})| \le \delta^2 \mu_J^- / 9, \tag{147}$$

eq:boundYJ

where $\mu_J^+ = \sum_{0\le i<m} \lfloor q_i \cdot |J\cap A_0|\rfloor p_i$ *and* $\mu_J^- = \sum_{0\le i<m} \lceil \tau_i q_i \cdot |J\cap A_0|\rceil p_i$*. Then for any* $J \in \Sigma_{s,\gamma}$*, we have*

$$\mathbb{P}(\neg\mathcal{T}_J \cap \mathfrak{X}^+_{\le m}) \le 3n^{-2w}.$$

Proof of Theorem 46 (assuming Lemma 47). The proof is similar to that of Theorem 33 in Section 3.2 via Lemma 34. By taking a union bound over all of the at most $|\Sigma_{s,\gamma}| \le n^w$ many configurations $J \in \Sigma_{s,\gamma}$, using Lemma 47 and $w = \omega(1)$ it follows that, with probability at $1 - n^{-\omega(1)}$, it follows that (146)–(147) hold for all $J \in \Sigma_{s,\gamma}$. We may assume $\delta \le 1$. By recursive definition of I_m we have

$$X_J - Y_J \le |J \cap I_m| \le X_J. \tag{148}$$ `eq:FIOinW`

Noting $\mu_J^- \ge \tau_m \mu_J^+ \ge (1-\delta/2)\mu_J^+$, Lemma 47 implies $X_J \le (1+\delta/2)\mu_J^+$ and

$$X_J - Y_J \ge (1-\delta/2-\delta^2/9)\cdot \mu_J^- \ge (1-\delta+\delta^2/8)\mu_J^+.$$

It thus suffices to show that $\mu_J^+ \sim \xi\Big((\log f)/D\Big)^{1/(r-1)}|J\cap A_0|$. But this is routine: as $J \in \Sigma_{s,\gamma}$, using $q_i|J\cap A_0|p_i \ge \frac{\sigma}{D^{\frac{1}{r-1}}}\gamma s^2/4 \gg p_i = \frac{\sigma}{q_i D^{\frac{1}{r-1}}}$ by (122) in Remark 43, we have

$$\begin{aligned}
\mu_J^+ &= \sum_{0\le i<m} (q_i|J\cap A_0| \pm 1)p_i \\
&\sim \sum_{0\le i<m} q_i|J\cap A_0| \cdot \frac{\sigma}{q_i D^{\frac{1}{r-1}}} = \frac{m\sigma}{D^{\frac{1}{r-1}}}|J\cap A_0| \qquad (149) \\
&\sim \xi\Big(\frac{\log f}{D}\Big)^{\frac{1}{r-1}}|J\cap A_0|,
\end{aligned}$$ `eq:mupinW`

which completes the proof of Theorem 46, as discused. □

Proof of Lemma 47. We start with $X_J = \sum_{0\le i<m} |J\cap A_i \cap \Gamma_{i+1}|$. Define

$$X^+_{J,i+1} := \mathbb{1}_{\{\mathfrak{X}_i\}} \sum_{v\in J\cap A_i} \mathbb{1}_{\{v\in\Gamma_{i+1}\}} \qquad \text{and} \qquad X_J^+ := \sum_{0\le i<m} X^+_{J,i+1}.$$

Note that $X_J = X_J^+$ when $\mathfrak{X}^+_{\le m} = \bigcap_{0\le i\le m}\mathfrak{X}_i^+$ holds. Let $Z^+_{J,i+1} \overset{\text{d}}{=} \text{Bin}(\lfloor q_i \cdot |J\cap A_0|\rfloor,\, p_i)$ be independent random variables. Since the $\mathcal{F}_i$-measurable event $\mathfrak{X}_i^+ \subseteq \mathcal{A}_i^+$ implies $|J\cap A_i| \le q_i\cdot|J\cap A_0|$, it is easy to see that $\mathbb{P}(X^+_{J,i+1} \ge t \mid \mathcal{F}_i) \le \mathbb{P}(Z^+_{J,i+1} \ge t)$ for $t \in \mathbb{R}$. Setting

$$Z_J^+ := \sum_{0\le i<m} Z^+_{J,i+1} \overset{\text{d}}{=} \sum_{0\le i<m} \text{Bin}\Big(\lfloor q_i \cdot |J\cap A_0|\rfloor,\, p_i\Big), \tag{150}$$ `eq:ZplusinW`

a standard stochastic domination argument then shows $\mathbb{P}(X_J^+ \ge t) \le \mathbb{P}(Z_J^+ \ge t)$ for $t \in \mathbb{R}$, so that

$$\mathbb{P}(X_J \ge t \text{ and } \mathfrak{X}_{\le m}) \le \mathbb{P}(X_J^+ \ge t) \le \mathbb{P}(Z_J^+ \ge t). \tag{151}$$ `eq:XUTinW`

Since $\mathfrak{X}_i^+ \subseteq \mathcal{A}_i^+$ also implies $|J\cap A_i| \ge \tau_i q_i \cdot |J\cap A_0|$, an analogous argument gives

$$\mathbb{P}(X_J \le t \text{ and } \mathfrak{X}^+_{\le m}) \le \mathbb{P}(Z_J^- \le t) \qquad \text{with} \qquad Z_J^- \overset{\text{d}}{=} \sum_{0\le i<m} \text{Bin}\Big(\lceil \tau_i q_i \cdot |J\cap A_0|\rceil,\, p_i\Big). \tag{152}$$ `eq:XLTinW`

Using $\mu_J^- \ge \frac{1}{2}\mu_J^+$, (149), $f \ge n^\eta$, $w = Ts = TB(\log n)^{1-\frac{1}{r-1}}D^{\frac{1}{r-1}}$ and $\frac{\delta^2\xi\gamma B\eta^{\frac{1}{r-1}}}{T} > 288$ by assumption of Theorem 41, for $J \in \Sigma_{s,\gamma}$ we have

$$\begin{aligned}
\delta^2 \min\{\mu_J^-, \mu_J^+\} &\ge \frac{\delta^2}{2}\mu_J^+ \\
&\ge \frac{\delta^2}{3}\xi\Big(\frac{\log f}{D}\Big)^{\frac{1}{r-1}}|J\cap A_0| \\
&\ge \frac{\delta^2}{3}\xi\Big(\frac{\eta\log n}{D}\Big)^{\frac{1}{r-1}}\frac{1}{4}\gamma B(\log n)^{1-\frac{1}{r-1}}D^{\frac{1}{r-1}}\cdot\frac{w}{T} > 24w\log n.
\end{aligned}$$

Since $\mathbb{E}Z_J^\pm = \mu_J^\pm$, by standard Chernoff bounds (see Remark 22) we obtain

$$\begin{aligned}\mathbb{P}\Big(X_J \notin [(1-\delta/2)\mu_J^-, (1+\delta/2)\mu_J^+] \text{ and } \mathfrak{X}_{\le m}^+\Big) &\le \mathbb{P}\Big(Z_J^- \le (1-\delta/2)\mu_J^-\Big) + \mathbb{P}\Big(Z_J^+ \ge (1+\delta/2)\mu_J^+\Big) \\ &\le \exp(-\delta^2\mu_J^-/8) + \exp(-\delta^2\mu_J^+/12) \\ &\le 2\exp(-2w\log n) = 2n^{-2w}.\end{aligned} \tag{153}$$

eq:XUTLYmuin

Finally, turning to $Y_J = \sum_{0\le i<m} |J\cap A_i \cap V(\mathcal{D}_{i+1})|$, for brevity we define

$$Y_{J,i+1} := |J\cap A_i\cap V(\mathcal{D}_{i+1})| \qquad \text{and} \qquad y_J := \delta^2\mu_J^-/9. \tag{154}$$

Note that $Y_J = \sum_{0\le i<m} Y_{J,i+1}$ and $Y_{J,i+1}\in\mathbb{N}$. Since $\mathfrak{X}_{\le i}^+ = \bigcap_{0\le j\le i}\mathfrak{X}_j^+$, a union bound argument gives

$$\begin{aligned}\mathbb{P}(Y_J \ge \delta^2\mu_J^-/9 \text{ and } \mathfrak{X}_{\le m}^+) &\le \sum_{\substack{(y_{J,1},\ldots,y_{J,m})\in\mathbb{N}^m\\ \sum_{0\le i<m} y_{J,i+1}=\lceil y_J\rceil}} \mathbb{P}\Big(\bigcap_{0\le i<m}\big(Y_{J,i+1}\ge y_{J,i+1} \text{ and } \mathfrak{X}_{\le i+1}^+\big)\Big) \\ &\le \sum_{\substack{(y_{J,1},\ldots,y_{J,m})\in\mathbb{N}^m\\ \sum_{0\le i<m} y_{J,i+1}=\lceil y_J\rceil}} \prod_{0\le i<m}\mathbb{P}\Big(Y_{J,i+1}\ge y_{J,i+1}\,\Big|\,\bigcap_{0\le j<i}\big(Y_{J,j+1}\ge y_{J,j+1} \text{ and } \mathfrak{X}_{\le j+1}^+\big)\Big).\end{aligned} \tag{155}$$

eq:sumboundi

Gearing up to apply Theorem 23 to $Y_{J,i+1}$, with an eye on $\mathcal{D}_{i+1}\subseteq B_{i+1}$ and $I_i\subseteq V_i$, we define (as mentioned, we identify the index $\alpha\in\mathcal{I}(J)$ with its first coordinate α_1 in this application) the index set

$$\mathcal{I}(J) := \bigcup_{2\le j\le r}\Big\{U\subseteq A_i:\ |U|=j,\ U\cap J\cap A_i\neq\varnothing,\ \text{there exists } \hat{U}\subseteq V_i,\ U\cup\hat{U}\in E_{\mathcal{H}}\Big\}.$$

For $\alpha\in\mathcal{I}(J)$, there exists a $v\in\alpha$ and $v\in J\cap A_i$, $\alpha\setminus\{v\}\in Y_{v,|\alpha|-1}(i)$, we infer by the usual reasoning ($\pi_i\ge\sigma$ and that $\mathfrak{X}_i^+\subseteq\mathcal{A}_i^+\cap\mathcal{Y}_i$ implies $|J\cap A_i|\le q_i|J\cap A_0|$ and $|Y_{v,j}(i)|\le\binom{r-1}{j}q_i^j\pi_i^{r-1-j}D^{\frac{j}{r-1}}$) that

$$\begin{aligned}\sum_{\alpha\in\mathcal{I}(J)}\mathbb{E}(\mathbb{1}_{\{\alpha\subseteq\Gamma_{i+1}\}}\mid\mathcal{F}_i) &\le \sum_{v\in J\cap A_i}\sum_{\alpha\in\mathcal{I}(J):v\in\alpha} p_i^{|\alpha|} \le \sum_{v\in J\cap A_i}\sum_{j=1}^{r-1}\sum_{U\in Y_{v,j}(i)} p_i^{|U\cup\{v\}|} \\ &\le q_i|J\cap A_0|\sum_{j=1}^{r-1}\binom{r-1}{j}q_i^j\pi_i^{r-1-j}D^{\frac{j}{r-1}}\Big(\frac{\sigma}{q_iD^{\frac{1}{r-1}}}\Big)^{j+1} \\ &= \frac{|J\cap A_0|}{D^{\frac{1}{r-1}}}\sum_{j=1}^{r-1}\binom{r-1}{j}\sigma^{j+1}\pi_i^{r-1-j} \\ &\le \frac{|J\cap A_0|}{D^{\frac{1}{r-1}}} r\cdot 2^r\sigma^2\pi_i^{r-2} =: \mu_{J,i+1}^*.\end{aligned}$$

Since $\mathcal{D}_{i+1}$ is a collection of vertex-disjoint elements of B_{i+1} (and thus $\{\hat{\alpha}\in\mathcal{D}_{i+1}:\alpha\cap\hat{\alpha}\neq\varnothing\}=\{\alpha\}$ for all $\alpha\in\mathcal{D}_{i+1}$), using $V(\mathcal{D}_{i+1})=\bigcup_{\alpha\in\mathcal{D}_{i+1}}\alpha\subseteq\Gamma_{i+1}\subseteq A_i$, $|\alpha|\le r$ and $I_i\subseteq V_i$, it is not difficult to check that

$$Y_{J,i+1} = \sum_{\alpha\in\mathcal{D}_{i+1}}|\alpha\cap J\cap A_i| \le r\cdot\sum_{\alpha\in\mathcal{I}(J)\cap\mathcal{D}_{i+1}}\mathbb{1}_{\{\alpha\subseteq\Gamma_{i+1}\}} \le rZ_1,$$

where Z_1 is defined as in Theorem 23. Applying (44) with $C=1$ and $\mu=\mu_{J,i+1}^*$ (in the probability space conditional on $\mathcal{F}_i$), whenever $\mathfrak{X}_{\le i}^+$ holds it follows that, say,

$$\mathbb{P}(Y_{J,i+1}\ge y_{J,i+1}\mid\mathcal{F}_i) \le \mathbb{P}(Z_1\ge\frac{y_{J,i+1}}{r}\mid\mathcal{F}_i) \le \begin{cases}\Big(\frac{e\mu_{J,i+1}^*}{y_{J,i+1}/r}\Big)^{y_{J,i+1}/r} \le \sigma^{y_{J,i+1}/(2r)} & \text{if } y_{J,i+1}\ge re\mu_{J,i+1}^*/\sqrt{\sigma}, \\ 1 & \text{otherwise.}\end{cases} \tag{156}$$

Boundeachter

Comparing the definition of $\sum_{0\le i<m}\mu^*_{J,i+1}$ with μ^-_J, using $\tau_i \ge \tau_m \ge \frac{1}{2}$ and $\sqrt{\sigma}\pi_i^{r-2} \ll \delta^2$ by (121) in Remark 43 we see that

$$\sum_{\substack{0\le i<m:\\ y_{J,i+1}\le re\mu^*_{J,i+1}/\sqrt{\sigma}}} y_{J,i+1} \le \frac{re}{\sqrt{\sigma}}\sum_{0\le i<m}\mu^*_{J,i+1} \ll \frac{\delta^2\mu^-_J}{9} = y_J.$$

So, inserting (156) into (155), using $\frac{y_J}{m\log y_J} > 1$ and $y_J = \Omega(w\log n)$, it follows that

$$\mathbb{P}(Y_J \ge \delta^2\mu^-_J/9 \text{ and } \mathfrak{X}^+_{\le m}) \le \sum_{\substack{(y_{J,1},\dots,y_{J,m})\in\mathbb{N}^m\\ \sum_{0\le i<m} y_{J,i+1}=\lceil y_J\rceil}} \sigma^{\frac{\lceil y_J\rceil}{2r}-o(y_J)} \le (y_J+2)^m\sigma^{\frac{\lceil y_J\rceil}{4r}} \ll n^{-2w},$$

completing the proof. □

sub:tYvi

4.3.3 Event $\tilde{\mathcal{Y}}_i$: crude bound on number $|\tilde{Y}_v(i)|$ of relaxed neighbors

In this subsection we prove estimate (124) via Lemma 48 below, which concerns a crude bound on number $|\tilde{Y}_v(i)|$ of relaxed neighbors defined in (107).

tY **Lemma 48.** *We have $\mathbb{P}(\neg\tilde{\mathcal{Y}}_i$ for some $0\le i\le m) \le n^{-\omega(1)}$.*

The following proof of Lemma 48 is very similar to the proof of Lemma 37 in Section 3.3, which concerned related (but slightly different) auxiliary variables.

Proof. Then case $r=2$ is trivial (since $\Delta_1 \le D$), so we may henceforth assume that $r\ge 3$. Note that when the coupling (91) from Section 3.3 holds, then for any vertex $v\in A_0$ we have

$$\tilde{Y}_v(i) \le \sum_{(U,e):e\in E_{\mathcal{H}},v\in e,U\subseteq e\setminus\{v\},|U|=r-2} \mathbb{1}_{\{U\subseteq V_i\}} \le \sum_{(U,e):e\in E_{\mathcal{H}},v\in e,U\subseteq e\setminus\{v\},|U|=r-2} \mathbb{1}_{\{U\subseteq V^+\}} =: \tilde{Y}^+_v, \qquad (157)$$ eq:def:tYv^+

recalling that V^+ is a P-random subset of A_0. For all $0\le i\le m$ it follows that

$$\mathbb{P}\Big(\tilde{Y}_v(i) \ge D^{\frac{1}{r-1}}Q^{r-1}\Big) \le \mathbb{P}\Big(\tilde{Y}^+_v \ge D^{\frac{1}{r-1}}Q^{r-1}\Big). \qquad (158)$$ eq:Lemma43

Gearing up to apply the upper tail inequality Theorem 23 to $\tilde{Y}^+_v$ via Remark 24 (exploiting the 'good' event $\mathcal{G}=\mathcal{N}^+$), note that

$$\mathbb{E}\tilde{Y}^+_v \le D\binom{r-1}{r-2}P^{r-2} \le rD^{\frac{1}{r-1}}Q^{r-2}.$$

Next we shall bound the parameter C from Remark 24 associated with $\tilde{Y}^+_v$. Using notation from Theorem 23 and Remark 24, we have $\xi(a) := \mathbb{1}_{\{a\in V^+\}}$ for vertices $a\in A_0$, and $\tilde{Y}^+_v = \sum_{\alpha\in\mathcal{I}} Y_\alpha$ for variables $Y_\alpha := \mathbb{1}_{\{U\subseteq V^+\}}$, where the index set $\mathcal{I}$ consists of all ordered pairs $\alpha=(U,e)$ that satisfy $e\in E_{\mathcal{H}}$, $v\in e$, $U\subseteq e\setminus\{v\}$, and $|U|=r-2$. We shall bound the associated C-term from Remark 24: to this end, given any $(U,e)\in\mathcal{I}$ with $U\subseteq V^+$, it suffices to bound the number of relevant $(\hat{U},\hat{e})\in\mathcal{I}$, i.e., which satisfy $U\cap\hat{U}\ne\varnothing$ and $\hat{U}\subseteq V^+$. For this we first derive one extra constraint. Since we require $\hat{U}\subseteq V^+$, for any $\hat{u}\in\hat{U}$, an edge $\hat{e}$ with $(\hat{U},\hat{e})\in\mathcal{I}$ must satisfy $\{\hat{u},v\}\subseteq\hat{e}$ and

$$|(\hat{e}\setminus\{\hat{u},v\})\cap V^+| \ge |\hat{U}\setminus\hat{u}| = r-3. \qquad (159)$$ eq:eextracon

With this extra constraint in hand, we are now ready to count the number of relevant $(\hat{U},\hat{e})\in\mathcal{I}$, as discussed. As we require $U\cap\hat{U}\ne\varnothing$, there are at most r ways to choose a vertex $\hat{u}\in U$ to be in the intersection. And then there are at most $\Delta^+_{2,r-3}$ (see (92) for definition of this parameter) edges $\hat{e}$ containing $\{\hat{u},v\}$ and satisfying (159). And then there are at most 2^r ways to choose $\hat{U}\subseteq\hat{e}$. It follows that, whenever the event $\mathcal{N}^+$ holds (which was defined in (93) of Section 3.3), the parameter C associated with $\tilde{Y}^+_v$ satisfies

$$C \le r\cdot\Delta^+_{2,r-3}\cdot 2^r \le r\cdot 2^r D^{\frac{1}{r-1}}f^{-1}Q^{r-2}.$$

Invoking Theorem 23 via Remark 24, using $Q = f^{2\xi^{r-1}}$ and $f \ge n^{\eta}$ we readily infer

$$\mathbb{P}(\tilde{Y}_v^+ \ge D^{\frac{1}{r-1}} Q^{r-1} \text{ and } \mathcal{N}^+) \le \left(\frac{1}{\Theta(Q)}\right)^{\frac{D^{\frac{1}{r-1}} Q^{r-1}}{C}} \le n^{-\omega(1)}, \tag{160}$$ eq:proftYv^+

which in view of inequality (158) and Lemma 36, completes the proof by a standard union bound argument (to account for all possible vertices $v \in A_0$). □

To sum up, Theorems 45–46 and Lemma 48 together complete the proof of Lemma 44 (and thus Theorem 41, which in turn implies Theorem 7, as discussed).

4.4 Strengthening of Theorem 7: proof of Corollary 8

oofofrandomH

By proceeding similarly to the way we strengthened Theorem 2 to Corollary 3 in Section 3.4, it is also straightforward to strengthen Theorem 7 to Corollary 8, the main idea being that $I \subseteq V_m$ is stochastically contained in a P-random vertex-subset of $V = V_{\mathcal{H}}$,

Proof of Corollary 8. The proof is very similar to the proof of Corollary 3: the only difference is that here we set $\xi := \min\{(\frac{\min\{\tau,\beta\}}{37})^{\frac{1}{r-1}}, (\frac{\zeta}{2})^{\frac{1}{r-1}}\}$ in the above proof of Theorem 41 (and thus Theorem 7), where the first requirement comes from (120) (to satisfy Remark 43), and the second ensures $n^{\zeta} \ge f^{\zeta} \ge f^{2\xi^{r-1}} = Q$ in (30). □

4.5 Verifying the smooth independence property

sub:vsi

The goal of this section is to prove the smooth independence result Lemma 39, which by definition (10) of the set of graphs $\mathfrak{F}$ is equivalent to the following more concrete lemma (that lists all relevant graphs explicitly).

oth:concrete **Lemma 49** (Smooth independence)**.** *The graph graph H has the smooth independence property if it is one of the following graphs:*

cliquesmooth (i) *a complete ℓ-vertex graph K_ℓ for fixed $\ell \ge 4$,*

cyclesmooth (ii) *a ℓ-cycle C_ℓ for fixed $\ell \ge 4$,*

artitesmooth (iii) *a complete bipartite graph $K_{a,a}$ for fixed $a \ge 4$, or*

ercubesmooth (iv) *a k-dimensional cube Q^k for fixed odd $k \ge 5$.*

The proof of this lemma is spread across the remainder of this section.

In Section 4.5, we set

$$p := n^{-\frac{v_H - 2}{e_H - 1}}. \tag{161}$$ eq:defofsmal

iscussionoff **Remark 50.** *As in the proof of Theorem 7, we can assume that in the r-uniform hypergraph $\mathcal{H}_H$ we have $r = e_H$, $D = \Theta(n^{v_H - 2})$, $f = n^{\eta}$ for some $\eta > 0$, and $p = \Theta(D^{-\frac{1}{r-1}})$ satisfying* (3)–(6) *and Lemma 18, for graphs H that we consider in Lemma 49.*

Let $\epsilon > 0$ be some small constant (specific requirements will be assigned in the proofs of (i)–(iv) in Lemma 49). As discussed around (91) there is a coupling satisfying $V_i \subseteq V^+$, where V^+ is a P-random vertex-subset of A_0, where each vertex $v \in A_0$ is included independently with probability $P \le pn^{\epsilon}$ by (30) and the choice of ξ, ϵ so that $f^{2\xi^{r-1}} \le n^{\epsilon}$. We have the following result regarding the inclusion probability.

offixedgraph **Lemma 51.** *For any fixed graph F on $[n]$ and all $0 \le i \le m$, we have $\mathbb{P}(F \subseteq V_i) \le P^{e_F} \le p^{e_F} n^{\epsilon e_F}$.*

4.5.1 Crude auxiliary bounds

:auxiboundsm

The conceptual crux is that Lemma 51 for the occurrence probability in V_i corresponds to the occurrence probability bound [8, Lemma 4.1] in the work of Bohman–Keevash on the H-free process, which means that we can reuse various estimates from their work that only rely on their Lemma 4.1 (in particular those from [8, Section 12]). We shall below collect a few such estimates.

First, [8, Lemma 4.1] is used to derive [8, Lemma 4.2], which in view of our Lemma 51 translates (as discussed) to the following result in our setting.

eenlargesets **Lemma 52.** *For any integers $a, b \geq 1$, with probability at least $1 - n^{-(a+b)}$, the following holds: for all $A, B \subseteq [n]$ with $|A| = a, |B| = b$,*

$$\max_{0 \leq i \leq m} |V_i \cap (A \times B)| \leq \max\big\{4\epsilon^{-1}(a+b),\ pabn^{2\epsilon}\big\},$$

where $A \times B := \big\{\{p_1, p_2\} :\ p_1 \in A,\ p_2 \in B\big\}$.

Second, [8, Lemma 4.2] is then used to derive [8, Lemma 4.3], which in view of our Lemma 51 translates (as discussed) to the following result in our setting.

BD:DAd **Lemma 53.** *For any integers a, d satisfying $16\epsilon^{-1} \leq d \leq a \leq \frac{1}{2}dp^{-1}n^{-2\epsilon}$, with probability at least $1 - n^{-a}$ the following holds: for all $A \subseteq [n]$ with $|A| = a$,*

$$\max_{0 \leq i \leq m} |D_{A,d}(i)| \leq 16\epsilon^{-1}d^{-1}a,$$

where $D_{A,d}(i) := \big\{p_1 \in [n] :\ \big|\{p_2 \in A :\ \{p_1, p_2\} \in V_i\}\big| \geq d\big\}$.

Similarly, in Section 5 of [8] the analogous version of Lemma 51 for H-free process is used to derive [8, Lemma 5.1], which in turn is used to derive [8, Lemma 5.2], which in turn is used to [8, Lemma 12.2]. Hence the following Lemmas 54–56 also hold in our setting.

For a graph G and a vertex-subset A of G, same as the definition in Section 1.2 of [8], recalling $p = n^{-\frac{v_H - 2}{e_H - 1}}$ in (161), we define

$$S_{A,G} := n^{v_G - |A|} p^{e_G - e_{G[A]}}. \tag{162}$$ def:SAG

Note that for $A \subseteq B \subseteq V_G$, $S_{B,G} = \frac{S_{A,G}}{S_{A,G[B]}}$.

We call (A, G) a *strictly balanced pair* if $S_{A,G} < S_{A,G[B]}$ for all $A \subsetneq B \subsetneq V_G$, or equivalently $S_{B,G} < 1$ for all $A \subsetneq B \subsetneq V_G$.

For a graph H, an *embedding* of H in V_i is an injection $f : V(H) \to [n]$ satisfying for all $\{p_1, p_2\} \in E_H$, $\{f(p_1), f(p_2)\} \in V_i$. Given a graph G, a vertex-subset A of G, and an injection $\phi : A \to [n]$, let $N_{\phi,G}(V_i)$ be the number of injections $g : V_G \to [n]$ satisfying $g|_A = \phi$ and for all $e \in E_G \setminus E_{G[A]}$, $g(e) \in V_i$.

ensions:easy **Lemma 54.** *Let $S^+_{B,G} := n^{v_G - |B|} P^{e_G - e_{G[B]}}$ for P as in (30). Then for any $\psi > 0$, with probability $1 - n^{-\omega(1)}$, for any injection $\phi : A \to [n]$ with $A \subseteq V_G$, we have*

$$N_{\phi,G}(V_i) \leq f^{\psi} \max_{A \subseteq B \subseteq V_G} S^+_{B,G}.$$

The following simple proof is inspired by arguments of Šileikis and Warnke [22] (see also [10]).

Proof. When the coupling (91) holds, it is enough to prove the result for $N_{\phi,G}(V^+)$, i.e., we replace V_i by V^+ in the definition of $N_{\phi,G}$. Fix an injection $\phi : A \to [n]$. It follows that

$$\mathbb{E}(N_{\phi,G})^{\kappa} \leq \sum_{(g_1,\dots,g_\kappa)} \mathbb{E}\Big(\prod_{i \in [\kappa]} \mathbb{1}_{\{g_i(E_G \setminus E_{G[A]}) \subseteq E^+\}}\Big) = \sum_{(g_1,\dots,g_\kappa)} P^{|\cup_{i\in[\kappa]} g_i(E_G \setminus E_{G[A]})|},$$

where the sum is over all κ-tuples $(g_1, \dots, g_\kappa)$ of injections g_i counted by $N_{\phi,G}$. Note that

$$g_i(E_G \setminus E_{G[A]}) \setminus \cup_{j \in [i-1]} g_j(E_G \setminus E_{G[A]}) \cong E_G \setminus E_{R_i}$$

and

$$g_i(E_G \setminus E_{G[A]}) \cap \big(\cup_{j\in[i-1]} g_j\big(E_G \setminus E_{G[A]}\big)\big) \cong E_{R_i} \setminus E_{G[A]}$$

for some $G[A] \subseteq R_i \subseteq G$. Taking all possible 'overlaps' $g_i(E_G \setminus E_{G[A]})$ with $\cup_{j\in[i-1]} g_j\big(E_G \setminus E_{G[A]}\big)$ into account, it follows that there are C_κ, D_κ (what may depend on A, G) such that

$$\mathbb{E}(N_{\phi,G})^\kappa \le \sum_{R_1,\dots,R_\kappa} D_\kappa \prod_{i\in[\kappa]} n^{v_G - v_{R_i}} P^{e_G - e_{R_i}} \le C_\kappa\big(\max_{G[A]\subseteq J\subseteq G} n^{v_G - v_J} P^{e_G - e_J}\big)^\kappa.$$

Noting that the maximum is attained by induced subgraphs, we infer

$$\max_{G[A]\subseteq J\subseteq G} n^{v_G - v_J} P^{e_G - e_J} = \max_{A\subseteq B\subseteq V_G} n^{v_G - |B|} P^{e_G - e_{G[B]}} = \max_{A\subseteq B\subseteq V_G} S^+_{B,G}.$$

It follows that for any $\psi > 0$, for $\kappa \ge \kappa_0(C_\kappa, \psi)$ and $n \ge n_0(C_\kappa)$ large enough, noting $f = n^\eta$, we have

$$\mathbb{P}(N_{\phi,G} \ge f^\psi \max_{A\subseteq B\subseteq V_G} S^+_{B,G}) \le C_\kappa f^{-\psi\kappa} \le f^{-\psi\kappa/2} \le n^{-(C+v_G+1)}.$$

After taking a union bound (to account for all injections ϕ), for all $n \ge n_0(C)$, the failure probability is less than n^{-C}. This completes the proof by the usual reasoning (since C was arbitrary). □

ensions:easy **Corollary 55.** *Assume that $\epsilon, \xi > 0$ satisfy $\eta\xi^{r-1} \le \epsilon$. Then, with probability $1 - n^{-\omega(1)}$, we have $N_{\phi,G}(V_i) \le n^{4e_G\epsilon} \max_{A\subseteq B\subseteq V_G} S_{B,G}$.*

Proof. We may assume $e_G \ge 1$, otherwise the result is trivially true. Note that $S^+_{B,G} \le Q^{e_G} \cdot S_{B,G}$. We then invoke Lemma 54 with $\psi := 2\xi^{r-1}$, which in view of $f^\psi Q^{e_G} = f^{\psi + 2e_G\xi^{r-1}} \le n^{\eta \cdot 4e_G\xi^{r-1}} \le n^{4e_G\epsilon}$ readily completes the proof. □

The next lemma offers a sufficient condition to verify if a graph H has smooth independence property, analogous to [8, Lemma 12.2].

lythensmooth **Lemma 56.** *Suppose that $(\{x,y\}, H\setminus\{u,v\})$ is strictly balanced for any two edges $\{u,v\}$, $\{x,y\}$ of H and H has minimum degree at least 3. Then H has smooth independence property.*

Proof. With the help of Corollary 55, which has the same form as Lemma 5.2 in [8], the proof of Lemma 12.2 in [8] carries over (noting that their vertex-set I and edge-set $P_I(i)$ play the roles of W and $B_{\binom{W}{2}}(i-1)$ in our setting, respectively). □

For a given vertex $p_1 \in [n]$, we define the set of neighbors and their number as

$$\mathcal{N}_{V_i}(p_1) := \big\{p_2 \in [n] :\ \{p_1, p_2\} \in V_i\big\} \qquad \text{and} \qquad N_{V_i}(p_1) := \big|\mathcal{N}_{V_i}(p_1)\big|.$$

Finally, we can bound the graph degree $N_{V_i}(p_1)$ by Lemma 51, an analog of which is used by Bohman–Keevash to prove [8, Lemma 12.1].

egreeingraph **Lemma 57.** *If the graph H satisfies $\frac{v_H-2}{e_H-1} \le 1$, then with probability at least $1 - n^{-\omega(1)}$, for all $p_1 \in [n]$ and $0 \le i \le m$,*

$$N_{V_i}(p_1) \le npn^{2\epsilon}.$$

Proof. By Lemma 51, for a fixed $p_1 \in [n]$, setting $x := npn^{2\epsilon}$, we have

$$\mathbb{P}\Big(|N_{V_i}(p_1)| \ge x\Big) \le \binom{n}{x} p^x n^{(1+o(1))\epsilon x} \le \Big(\frac{enpn^{(1+o(1))\epsilon}}{x}\Big)^x \le n^{-\frac{1}{2}\epsilon x} \le n^{-\omega(1)}.$$

We complete the proof by taking a union bound over all $p_1 \in [n]$. □

4.5.2 Proof of smooth independence result Lemma 49

smooth:indep

With the auxiliary estimates from Section 4.5.1 in hand, we are now ready to verify the smooth independence property of the graphs listed in Lemma 49. We first focus on cycles C_ℓ with $\ell \ge 5$, where the path counting argument from [8, Section 12] carries over.

Proof of Lemma 49 (ii). With an eye on Remark 50, we set $\eta\tau = 5\ell\epsilon$ so that $f^\tau = n^{5\ell\epsilon}$ (recalling we have the freedom to choose τ and ϵ). The proof follows that of [8, Lemma 12.1] by recalling their definition of smooth independence and the size parameter α in [8, Section 11] and making few minor changes as below. To avoid clutter of I in their proof with our notation of independent set in hypergraph, we replace their I with W. We define

$$W_j := \{v : |\mathcal{N}_{V_i}(v) \cap W_{j-1}| > n^{-10\ell\epsilon} pn\},$$

and use Lemma 57 to bound the degree of any vertex. With ϵ much less than $1/\ell$ in mind, i.e. $10\ell(\ell-1)\epsilon < 1/(\ell-1)$, their robust proof then carries over. □

We next turn to cliques K_ℓ with $\ell \ge 5$, where the density argument from [8, Section 12] carries over.

Proof of Lemma 49 (i) *for* $\ell \ge 5$. When $H = K_\ell$, we have $r = \binom{\ell}{2}$, therefore $p = n^{-\frac{\ell-2}{r-1}} = n^{-\frac{2}{\ell+1}}$. When $\ell \ge 6$, for any distinct edges $\{x,y\},\{u,v\} \in E_H$ and vertex-set B satisfying $\{x,y\} \subsetneq B \subsetneq V_H$, it is easy to check that $S_{\{x,y\},(H\setminus\{u,v\})[B]} > S_{\{x,y\},H\setminus\{u,v\}}$, therefore $S_{B,H\setminus\{u,v\}} = S_{\{x,y\},H\setminus\{u,v\}}/S_{\{x,y\},(H\setminus\{u,v\})[B]} < 1$, i.e., $(\{x,y\}, H \setminus \{u,v\})$ is strictly balanced. Then Lemma (i) for $\ell \ge 6$ cases follows from Lemma 56. When $\ell = 5$, the proof of Lemma 12.3 in [8] carries over (with the same observation as in the proof of Lemma 56). □

Verifying smooth independence of K_4 is conceptually similar but more involved, and we thus defer the proof of Lemma 49 (i) for $\ell = 4$ to Appendix A.2.1. The proofs of Lemma 49 (iii)–(iv) for complete bipartite graphs $K_{a,a}$ and k-dimensional cubes Q^k are straightforward applications of Lemma 56, and we defer these routine arguments (that involve a number of technical estimates that are rather tangential to the main arguments here) to Appendices A.2.2–A.2.3. This completes the proof of Lemma 49 (and thus Lemma 39, as discussed).

5 Pseudo-randomness: vertex-distribution

seudorandom1

The goal of this section is to prove the pseudo-random result Theorem 5 and its consequence Corollary 6, i.e., to show that the semi-random greedy independent set algorithm constructs an independent set $I = I_m$ that intuitively 'looks like' a random vertex-subset of $V_{\mathcal{H}}$ (where each vertex is included independently with a certain probability). To this end we shall prove the following generalization of Theorem 5, which holds for every intermediate independent set I_i of the semi-random algorithm (not just for the final set $I = I_m$).

edgesofGinIi **Theorem 58.** *Fix $r \ge 2$ and $L \ge 1$. Let $\mathcal{H} = (V,E)$ be an r-uniform hypergraph satisfying the assumptions of Theorem 2. Then, for all $W \subseteq V$ of size $|W| \le L$ that do not contain an edge of $\mathcal{H}$, for all steps $0 \le i \le m$ we have*

$$\mathbb{P}(W \subseteq I_i) \;=\; (1+o(1))\varrho_i^{|W|} \quad \textit{with} \quad \varrho_i := \frac{\sigma i}{D^{\frac{1}{r-1}}}, \tag{163}$$ eq:probofXG

where $\sigma = \sigma(f,r) > 0$ is defined as in (14) *and $m = m(\xi,d,D,r)$ is defined as in* (15).

Using the pseudo-random estimate (163), a routine second moment calculation yields the following generalization of Corollary 6, which again holds for every intermediate independent set I_i (not just for $I = I_m$). Let $X_{\mathcal{G}}(i) := |E(\mathcal{G}[I_i])|$ denote the number of edges in $\mathcal{G}$ that are completely contained in the independent set I_i constructed by the semi-random independent set algorithm.

ndom:general **Corollary 59.** *Fix $r,t \ge 2$. Let $\mathcal{H} = (V,E)$ be an r-uniform hypergraph satisfying the assumption of Theorem 2. Let $\mathcal{G}$ be a t-uniform hypergraph on the vertex-set V, such that no edge of $\mathcal{G}$ contains an edge of $\mathcal{H}$. If $|E(\mathcal{G})| \cdot \varrho_i^t \to \infty$ and $\Delta_a(\mathcal{G}) = o\big(|E(\mathcal{G})| \cdot \varrho_i^a\big)$ for $1 \le a \le t-1$, then with high probability*

$$X_{\mathcal{G}}(i) = (1+o(1))|E(\mathcal{G})|\varrho_i^t, \tag{164}$$ eq:cor:pseud

where $\varrho_i = \varrho_i(f,D,r) > 0$ is defined as in (163).

Proof of Corollary 59 (assuming Theorem 58). Applying Theorem 58 with $L = 25$, we may henceforth assume that (163) holds for all $1 \le \kappa \le 2t$. The first moment of $X_{\mathcal{G}}(i)$ therefore satisfies

$$\mathbb{E}X_{\mathcal{G}}(i) = |E(\mathcal{G})| \cdot (1+o(1))\varrho_i^t \to \infty. \tag{165}$$

eq:cor:pseud

Turning to the second moment of $X_{\mathcal{G}}(i)$, note that

$$\mathbb{E}X_{\mathcal{G}}(i)^2 = \sum_{f_1 \in E(\mathcal{G})} \sum_{f_2 \in E(\mathcal{G})} \mathbb{P}(f_1 \cup f_2 \subseteq I_i). \tag{166}$$

eq:cor:pseud

The crux is now that the upper bound

$$\mathbb{P}(f_1 \cup f_2 \subseteq I_i) \;\le\; (1+o(1))\varrho_i^{|f_1 \cup f_2|} \tag{167}$$

eq:twoedgeso

follows by a case distinction: (i) if $f_1 \cup f_2$ does not contain an edge of $\mathcal{H}$, then (167) holds with equality by (163), and (ii) if $f_1 \cup f_2$ contains an edge of $\mathcal{H}$, then the probability in (167) equals zero since the independent set I_i contains no edges of $\mathcal{H}$ (by construction). Taking into accounts all possible overlaps of any two hyperedges $f_1, f_2 \in E(\mathcal{G})$, using $\Delta_t(\mathcal{G}) \le 1$ and $t = O(1)$ together with the assumption that $\Delta_a(\mathcal{G}) = o\big(|E(\mathcal{G})| \cdot \varrho_i^a\big)$ for $1 \le a \le t-1$, by combining (166) with estimates (167) and (165) it follows that

$$\mathbb{E}X_{\mathcal{G}}(i)^2 \le |E(\mathcal{G})|^2 \cdot (1+o(1))\varrho_i^{2t} + O\bigg(\sum_{1 \le a \le t} |E(\mathcal{G})| \cdot \Delta_a(\mathcal{G}) \cdot \varrho_i^{2t-a}\bigg) = (1+o(1))\big(\mathbb{E}X_{\mathcal{G}}(i)\big)^2,$$

which readily completes the proof of the with high probability estimate (164) by the second moment method (i.e., an application of Chebyshev's inequality). □

In the upcoming proof of Theorem 58 it will be useful to complement the supplement the 'closure probability estimate' (55) from Lemma 29 with the lower bound in (168) below.

tiprobnotinC **Lemma 60.** *Fix $L \ge 1$. For all $W \subseteq A_i$ with $|W| \le L$, we have*

$$(1 - \sigma^{1+\rho/2}\pi_i^{\max\{r-3,0\}})\Big(\frac{q_{i+1}}{q_i}\Big)^{|W|} \le \mathbb{P}(W \cap C_{i+1} = \varnothing \mid \mathfrak{X}_{\le i}) \le \Big(\frac{q_{i+1}}{q_i}\Big)^{|W|}. \tag{168}$$

eq:lemma:gen

Proof of Lemma 60. The proof is very similar to that of Lemma 29 (using $|Y_{w_1} \cup \cdots \cup Y_{w_j}| \le \sum_{1 \le \ell \le j} |Y_{w_\ell}|$ for the lower bound, and the term $\sigma^{\rho/2}$ to absorb constants), and we thus leave the straightforward details to the reader. □

Proof of Theorem 58. For $i = 0$ the statement is trivially true, due to $I_0 = \varnothing$ and the standard convention $0^0 = 1$. In the following, we may therefore assume $i \ge 1$. Suppose $W = \{v_1, \ldots, v_\kappa\}$. Since $\mathbb{P}(\neg\mathfrak{X}_{\le i}) = N^{-\omega(1)}$ is negligible compared to the main term in the formula, it suffices to estimate the probability of

$$\mathbb{P}(v_1, \ldots, v_\kappa \in I_i \text{ and } \mathfrak{X}_{\le i}). \tag{169}$$

eq:aimevent

To estimate this probability, the idea is to sum over all i^κ choices of the steps $1 \le i_j \le i$ where the vertex v_j is included into the independent set I_{i_j} from $I_{i_j - 1}$ (to clarify: here the steps i_j of the vertices do not need to be distinct, since multiple vertices can be added in each step of the semi-random construction). Hence for each choice of $(i_1, \ldots, i_\kappa)$ there is $1 \le \tilde{\kappa} \le \kappa$ and steps $1 \le m_1 < \cdots < m_{\tilde{\kappa}} \le i$ and non-empty disjoint sets $J_1, \ldots, J_{\tilde{\kappa}}$ satisfying $J_1 \cup \cdots \cup J_{\tilde{\kappa}} = \{v_1, \ldots, v_\kappa\}$ such that, for each $1 \le p \le \tilde{\kappa}$, the vertices in J_p are all included into the independent set I_{m_p} from $I_{m_p - 1}$. To facilitate working with these definitions, set

$$\mathcal{E} = \mathcal{E}(m_1, \ldots, m_{\tilde{\kappa}}, J_1, \ldots, J_{\tilde{\kappa}}) := \bigcap_{1 \le p \le \tilde{\kappa}} \{J_p \subseteq \Gamma_{m_p} \setminus V(\mathcal{D}_{m_p})\} \cap \mathfrak{X}_{\le i}. \tag{170}$$

eq:def:cEv

Let $\mathcal{E}_j$ be the event that the first j steps are compatible with the event $\mathcal{E}$. Then an upper bound on (169) is

$$\mathbb{P}(\mathcal{E}) \le \prod_{1 \le \ell \le m_{\tilde{\kappa}}} \mathbb{P}(\mathcal{E}_\ell \mid \mathcal{E}_{\ell-1}).$$

For $1 \le \ell < m_{\tilde{\kappa}}$, let $h(\ell)$ be the smallest p so that $\ell < m_p$. When $\ell = m_p$ for some $1 \le p \le \tilde{\kappa}$, then

$$\mathbb{P}(\mathcal{E}_\ell \mid \mathcal{E}_{\ell-1}) \le \mathbb{P}(J_p \subseteq \Gamma_{m_p} \mid \mathcal{E}_{m_p-1}) = \left(\frac{\sigma}{q_{m_p-1}D^{\frac{1}{r-1}}}\right)^{|J_p|}.$$

Otherwise, when $\ell \neq m_p$ for all $1 \le p \le \tilde{\kappa}$, then together with Lemma 60,

$$\mathbb{P}(\mathcal{E}_\ell \mid \mathcal{E}_{\ell-1}) \le \mathbb{P}\Big((\cup_{h(\ell)\le q\le\tilde{\kappa}} J_q) \cap C_\ell = \varnothing \mid \mathcal{E}_{\ell-1}\Big) \le \Big(\frac{q_\ell}{q_{\ell-1}}\Big)^{\sum_{h(\ell)\le q\le\tilde{\kappa}}|J_q|}.$$

Multiplying these probabilities, and noting that most terms cancel, using $q_0 = 1$, $q_{\ell-1}/q_\ell \le 1 + O(\sigma^{1/2})$ (see Remark 28), $\sum_{p=1}^{\tilde{\kappa}} |J_p| = \kappa$, and $\tilde{\kappa} \le \kappa$, it follows that, say

$$\mathbb{P}(\mathcal{E}) \le \big(1 + O(\sigma^{1/2})\big)^{\kappa^2} \big(\frac{\sigma}{D^{\frac{1}{r-1}}}\big)^{\kappa} = (1 + o(1))\big(\frac{\sigma}{D^{\frac{1}{r-1}}}\big)^{\kappa}. \tag{171}$$ eq:aimevent:

Putting things together, this establishes the upper bound in (169) by summing the probability (171) over all the i^κ possible choices of the vertex-inclusion sequences $(i_1, \ldots, i_\kappa)$.

The proof of the lower bound in (169) is conceptually similar, but we need more involved events to guarantee that certain vertices are added; we defer the more technical details to Appendix A.3. □

6 Final remarks: weaker assumptions

We close by remarking that, with some care in the proofs, we can weaken some of the technical assumptions used in this paper. For example, Theorem 2 and Corollary 3 are still true if we relax the definition of Γ in (2) by restricting to the pairs of vertices v, v' that are contained in an edge e of $\mathcal{H}$, i.e., replacing assumption (5) by

$$\Gamma_e(\mathcal{H}) \le D/f, \tag{172}$$ assumption3n

where the definition of $\Gamma_e(\mathcal{H})$ is the same as that of $\Gamma(\mathcal{H})$ from (2), except that we now insist on the two vertices v, v' being contained in some edge of $\mathcal{H}$. Note that in the graph case $r = 2$ (which is not our main focus) the relaxed parameter $\Gamma_e(\mathcal{H})$ counts the maximum number of triangles containing any edge, which means that (172) is reminiscent of the triangle-free condition used, e.g., by Ajtai–Komlós–Szemerédi [2] and Shearer [21] in the analysis of the independence number. In the graph case $r = 2$ of Theorem 2 and Corollary 3, we can alternatively also replace assumption (5) by

$$\max_{v\in V_{\mathcal{H}}} |\{E_C \subseteq E_{\mathcal{H}} : C \cong C_4,\ v \in V_C\}| \le D^3/f, \tag{173}$$ eq:numberofC

where C_4 denotes the 4-vertex cycle, as usual. Note that in this case the left-hand side of (173) counts the maximum number of 4-vertex cycles containing any vertex, which means that (173) is reminiscent of the 'girth is at least 5' condition used, e.g., by Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] and Kim [17] in the analysis of the independence and chromatic number, respectively.

Finally, the discussion above raises the intriguing question whether assumption (5) is needed in Theorem 2 and Corollary 3, i.e., if the semi-random greedy independent set algorithm still constructs an independent set $I \subseteq V$ of size (7) if assumption (5) is dropped. Perhaps rashly, we speculate that in general some kind of extra assumption is in fact needed (which does not rule out that assumption (5) can perhaps be relaxed if one modifies the semi-random construction further).

Acknowledgements. We thank Dhruv Mubayi for encouraging us to include the pseudo-random properties stated in Section 1.2.1.

References

1982extremal [1] Miklós Ajtai, János Komlós, Janos Pintz, Joel Spencer, and Endre Szemerédi. Extremal uncrowded hypergraphs. *Journal of Combinatorial Theory, Series A*, 32(3):321–335, 1982.

denseinfinte [2] Miklós Ajtai, János Komlós, and Endre Szemerédi. A dense infinite Sidon sequence. *European J. Combin.*, 2(1):1–11, 1981.

018bipartite [3] Deepak Bal and Patrick Bennett. The bipartite $K_{2,2}$-free process and bipartite Ramsey number $b(2,t)$. *Electron. J. Combin.*, 27(4):Paper No. 4.23, 13, 2020.

4Independent [4] József Balogh, Robert Morris, and Wojciech Samotij. Independent sets in hypergraphs. *Journal of the American Mathematical Society*, 28(3):539–550, 2014.

B2020-SF [5] Patrick Bennett. The sum-free process. *Electron. J. Combin.*, 27(1):Paper No. 1.13, 18, 2020.

nett2016note [6] Patrick Bennett and Tom Bohman. A note on the random greedy independent set algorithm. *Random Structures & Algorithms*, 49(3):479–502, 2016.

BP2020 [7] Patrick Bennett and Andrzej Dudek. A gentle introduction to the differential equation method and dynamic concentration. *Discrete Math.*, 345(12):Paper No. 113071, 17, 2022.

man2010early [8] Tom Bohman and Peter Keevash. The early evolution of the H-free process. *Inventiones mathematicae*, 181(2):291–336, 2010.

n2013dynamic [9] Tom Bohman and Peter Keevash. Dynamic concentration of the triangle-free process. *Random Structures Algorithms*, 58(2):221–293, 2021.

man2018large [10] Tom Bohman and Lutz Warnke. Large girth approximate Steiner triple systems. *Journal of the London Mathematical Society*, 100(3):895–913, 2019.

01asymptotic [11] Yair Caro and Cecil Rousseau. Asymptotic bounds for bipartite Ramsey numbers. *the electronic journal of combinatorics*, 8(1):17, 2001.

g1988induced [12] Fan RK Chung, Zoltán Füredi, Ronald L Graham, and Paul Seymour. On induced subgraphs of the cube. *Journal of Combinatorial Theory, Series A*, 49(1):180–187, 1988.

2016coloring [13] Jeff Cooper and Dhruv Mubayi. Coloring sparse hypergraphs. *SIAM Journal on Discrete Mathematics*, 30(2):1165–1180, 2016.

a.3240060208 [14] Richard A. Duke, Hanno Lefmann, and Vojtech Rödl. On uncrowded hypergraphs. *Random Structures & Algorithms*, 6(2-3):209–212, 1995.

o2017packing [15] He Guo and Lutz Warnke. Packing nearly optimal Ramsey $R(3,t)$ graphs. *Combinatorica*, 40(1):63–103, 2020.

GW2022-AP [16] He Guo and Lutz Warnke. On the power of random greedy algorithms. *European J. Combin.*, 105:Paper No. 103551, 15, 2022.

im1995brooks [17] Jeong Han Kim. On brooks' theorem for sparse graphs. *Combinatorics, Probability and Computing*, 4(2):97–132, 1995.

P2014-D [18] Michael E. Picollelli. The diamond-free process. *Random Structures Algorithms*, 45(3):513–551, 2014.

P2014-Cl [19] Michael E. Picollelli. The final size of the C_ℓ-free process. *SIAM J. Discrete Math.*, 28(3):1276–1305, 2014.

15Hypergraph [20] David Saxton and Andrew Thomason. Hypergraph containers. *Inventiones Mathematicae*, 201(3):925–992, 2015.

trianglefree [21] James B. Shearer. A note on the independence number of triangle-free graphs. *Discrete Math.*, 46(1):83–87, 1983.

2018counting [22] Matas Šileikis and Lutz Warnke. Counting extensions revisited. *Random Structures Algorithms*, 61(1):3–30, 2022.

Szemeredikap [23] E. Szemerédi. On sets of integers containing no k elements in arithmetic progression. *Acta Arith.*, 27:199–245, 1975.

W2014-Cl [24] Lutz Warnke. The C_ℓ-free process. *Random Structures Algorithms*, 44(4):490–526, 2014.

W2014-K4 [25] Lutz Warnke. When does the K_4-free process stop? *Random Structures Algorithms*, 44(3):355–397, 2014.

Warnke2017 [26] Lutz Warnke. Upper tails for arithmetic progressions in random subsets. *Israel Journal of Mathematics*, 221(1):317–365, 2017.

W2019 [27] Lutz Warnke. On Wormald's differential equation method. *arXiv e-prints*, page arXiv:1905.08928, May 2019.

W1995 [28] Nicholas C. Wormald. Differential equations for random processes and random graphs. *Ann. Appl. Probab.*, 5(4):1217–1235, 1995.

A Appendix: deferred proofs

This appendix only appears in the arXiv version of this paper: it contains some technical proofs deferred from Section 1.1, Section 4.5.2 and Section 5 (which are somewhat tangential to the main arguments of the paper).

A.1 Auxiliary hypergraph for strictly 2-balanced graphs H (Section 1.1)

ryhypergraph

Proof of Lemma 4. It suffices to prove the claim for $\mathcal{H} = \mathcal{H}_H$. Given any edges, the maximum number of H-copies containing that edge is at most $O(n^{v_H-2})$. Hence the maximum 1-degree of $\mathcal{H}$ readily satisfies

$$\Delta_a \le A n^{v_H-2}$$

for some positive constant $A > 0$, establishing assumption (3).

To bound Δ_a in assumption (4) from above, given any a edges for $2 \le a \le r-1 = e_H - 1$ (a vertices in hypergraph setting), we shall bound the number of ways to extend them to a copy of H. Let $H_0 \subseteq H$ be a subgraph of H with a edges and minimum number of vertices. As $2 \le a \le r-1$, we have $H_0 \subsetneq H$ and $v_{H_0} \ge 3$. Note that

$$\Delta_a = O(n^{v_H - v_{H_0}}) \quad \text{and} \quad D^{\frac{r-a}{r-1}} = \Theta\Big(n^{(v_H-2)\frac{e_H-e_{H_0}}{e_H-1}}\Big)$$

Since H is strictly 2-balanced, it follows that $(v_H-2)\frac{e_H-e_{H_0}}{e_H-1} > v_H - v_{H_0}$. Hence there exists a constant $\eta \in (0,1)$ such that, setting $f = n^{\eta}$, we have

$$\Delta_a \le D^{\frac{r-a}{r-1}}/f,$$

establishing assumption (4).

Since the minimum degree is at least two, we also see that

$$\Gamma(\mathcal{H}) = O(n^{v_H-2-1}) \le D/f$$

by choice of $f = n^{\eta}$ with $\eta \in (0,1)$, establishing assumption (5).

Finally, the verification of (6) is straightforward and left to the reader. □

A.2 Smooth independence (Section 4.5.2)

quesmoothetc

In this appendix we establish that several graphs have the smooth independence property, giving proofs that were deferred from Section 4.5.2. The arguments are elementary and often boil down to verifying certain technical inequalities (that are not particularly insightful, which is why they are deferred to this appendix).

A.2.1 Complete graph K_4: proof of Lemma 49 (i) for $\ell = 4$

apx:K4

Proof of Lemma 49 (i) *for* $\ell = 4$. The general rule for τ and ϵ in this proof is that f^{τ} and n^{ϵ} are tiny compared to D, say $f^{2\tau} n^{30\epsilon} (\log n)^9 \ll D^{\frac{1}{2(r-1)}}$. Let us first discuss how we shall verify the two bounds (109) and (110). In concrete words, $B_{\binom{W}{2}}(i)$ defined in Section 4.2 here simply consists of all pairs $g_1 = \{c,d\} \in A_0$ with the following property: there are at least $f^{-\tau} D^{\frac{1}{r-1}}$ many pairs $g_2 = \{x,y\} \in \binom{W}{2}$ and $g_1 \neq g_2$ for which adding both pairs g_1, g_2 to the edge-set V_i creates a copy of K_4 containing both g_1, g_2 as edges. Motivated by the fact that the pair g_2 is either disjoint from g_1 or not, we then introduce the following two sets

$$B^{(1)}_{\binom{W}{2}}(i) := \Big\{ \{c,d\} \in A_0 \,:\, \text{there are} \ge \frac{1}{6} f^{-\tau} D^{\frac{1}{r-1}} \text{ many } \{u,v\} \in \binom{W}{2} \text{ with } \{u,v\} \cap \{c,d\} = \varnothing \text{ such that } \binom{\{c,d,u,v\}}{2} \setminus (\{c,d\},\{u,v\}) \subseteq V_i \Big\},$$

$$B^{(2)}_{\binom{W}{2}}(i) := \Big\{ \{c,d\} \in A_0 \,:\, \text{there are} \ge \frac{1}{6} f^{-\tau} D^{\frac{1}{r-1}} \text{ many } \{d,u\} \in \binom{W}{2} \text{ with } \{d,u\} \cap \{c,d\} = \{d\} \text{ for which there exists } v \notin \{c,d,u\} \text{ with } \binom{\{c,d,u,v\}}{2} \setminus (\{c,d\},\{d,u\}) \subseteq V_i \Big\}, \tag{174}$$

eq:def:BW2

which by definition satisfy $B_{\binom{W}{2}}(i) \subseteq B^{(1)}_{\binom{W}{2}}(i) \cup B^{(2)}_{\binom{W}{2}}(i)$. To verify (109), it thus is enough to show that

$$\max_{j\in\{1,2\}} \big|B^{(j)}_{\binom{W}{2}}(i)\big| \le \frac{1}{2} f^{-2\tau} D^{\frac{2}{r-1}}. \tag{175}$$

eq:K4:BW:goa

Turning to $P_{\binom{W}{2}}(i)$ defined in Section 4.2, for $j \in \{1,2\}$ we similarly introduce the following two sets

$$P^{(j)}_{\binom{W}{2}}(i) := B^{(j)}_{\binom{W}{2}}(i) \cap V_i,$$

which by definition satisfy $P_{\binom{W}{2}}(i) \subseteq P^{(1)}_{\binom{W}{2}}(i) \cup P^{(2)}_{\binom{W}{2}}(i)$. To verify (110), it then is enough to show that

$$\max_{j\in\{1,2\}} \big|P^{(j)}_{\binom{W}{2}}(i)\big| \le \frac{1}{2} f^{-\tau} D^{\frac{1}{r-1}}. \tag{176}$$

eq:K4:PW:goa

We start by verifying the case $j = 1$ of both (175) and (176). To this end, we introduce

$$T_{W,1} := \Big\{c \;:\; \{c,d\} \in B^{(1)}_{\binom{W}{2}}(i) \text{ for some } d\Big\}.$$

Then, for any $c \in T_{W,1}$ and $\{c,d\} \in B^{(1)}_{\binom{W}{2}}(i)$, we have

$$|\mathcal{N}_{V_i}(c) \cap W| \ge |\mathcal{N}_{V_i}(c) \cap \mathcal{N}_{V_i}(d) \cap W| \ge \frac{1}{\sqrt{6}} f^{-\frac{\tau}{2}} D^{\frac{1}{2(r-1)}},$$

where the second inequality holds by definition of $B^{(1)}_{\binom{W}{2}}(i)$, since otherwise $\{c,d\}$ could form less than $(\frac{1}{\sqrt{6}} f^{-\frac{\tau}{2}} D^{\frac{1}{2(r-1)}})^2 = \frac{1}{6} f^{-\tau} D^{\frac{1}{r-1}}$ copies of K_4 with pairs $\{u,v\}$ that are disjoint from $\{c,d\}$. Therefore we arrive at the lower bound

$$|V_i \cap (T_{W,1} \times W)| \ge \frac{1}{2} \sum_{c\in T_{W,1}} |\mathcal{N}_{V_i}(c) \cap W| \ge \frac{1}{2} \cdot \frac{1}{\sqrt{6}} f^{-\frac{\tau}{2}} D^{\frac{1}{2(r-1)}} \cdot |T_{W,1}|. \tag{177}$$

eq:K4:loweru

Using Lemma 52 we can also get an upper bound on $|V_i \cap (T_{W,1} \times W)|$ that holds with probability $1 - n^{-\omega(1)}$ for all such W simultaneously, leading to

$$\frac{1}{2\sqrt{6}} f^{-\frac{\tau}{2}} D^{\frac{1}{2(r-1)}} |T_{W,1}| \;\le\; |V_i \cap (T_{W,1} \times W)| \;\le\; \max\big\{4\epsilon^{-1}(|T_{W,1}| + |W|),\; p|T_{W,1}| \cdot |W| n^{2\epsilon}\big\}. \tag{178}$$

eq:K4:loweru

Noting that $|W| = \Theta\Big((\log n)^{1-\frac{1}{r-1}} D^{\frac{1}{r-1}}\Big)$ and $p = \Theta(D^{-\frac{1}{r-1}})$, it is easy to check that the term on the left-most side of (178) is much larger than $p|T_{W,1}| \cdot |W| n^{2\epsilon}$, so only the first term in the maximum on the right-hand side of (178) matters. Since $f^{-\frac{\tau}{2}} D^{\frac{1}{2(r-1)}} \gg 1$, it thus follows that (178) implies, say,

$$\frac{1}{12} f^{-\frac{\tau}{2}} D^{\frac{1}{2(r-1)}} |T_{W,1}| \le 4\epsilon^{-1}|W|, \tag{179}$$

eq:K4:loweru

which together with $|W| = \Theta\Big((\log n)^{1-\frac{1}{r-1}} D^{\frac{1}{r-1}}\Big)$ implies

$$|T_{W,1}| = O\Big((\log n)^{1-\frac{1}{r-1}} f^{\frac{\tau}{2}} D^{\frac{1}{2(r-1)}}\Big). \tag{180}$$

eq:K4:bound:

It then readily follows that

$$|B^{(1)}_{\binom{W}{2}}(i)| \le |T_{W,1}|^2 \le \Theta\Big((\log n)^{2-\frac{2}{r-1}} f^{\tau} D^{\frac{1}{r-1}}\Big) \ll \frac{1}{2} f^{-2\tau} D^{\frac{2}{r-1}}, \tag{181}$$

BW1

establishing (175) for $j = 1$. Recalling that $p = \Theta(D^{-\frac{1}{r-1}})$, using Lemma 52 we also obtain that, with probability $1 - n^{-\omega(1)}$, for all relevant sets W we have

$$\begin{aligned} |P^{(1)}_{\binom{W}{2}}(i)| &\le |V_i \cap (T_{W,1} \times T_{W,1})| \le \max\bigl\{8\epsilon^{-1}|T_{W,1}|,\ p \cdot |T_{W,1}|^2 \cdot n^{2\epsilon}\bigr\} \\ &\le \Theta\Bigl((\log n)^{1-\frac{1}{r-1}} f^{\frac{\tau}{2}} D^{\frac{1}{2(r-1)}}\Bigr) \ll \frac{1}{2} f^{-\tau} D^{\frac{1}{r-1}}, \end{aligned} \tag{182}$$

PW1

establishing (176) for $j = 1$.

It thus remains to verify the case $j = 2$ of both (175) and (176). For $B^{(2)}_{\binom{W}{2}}(i)$, let $T_{W,2}$ be the collection of all c as the notation in (174), that is

$$T_{W,2} := \Bigl\{c : \text{there exists } d \text{ such that } \{c,d\} \in B^{(2)}_{\binom{W}{2}}(i)\Bigr\}.$$

Note that in the above definition, we have $\{c,u\} \in V_i$, which implies $|\mathcal{N}_{V_i}(c) \cap W| \ge \frac{1}{6} f^{-\tau} D^{\frac{1}{r-1}}$. Proceeding similarly to (177), we thus obtain

$$|V_i \cap (T_{W,2} \times W)| \ge \frac{1}{2} \sum_{c \in T_{W,2}} |\mathcal{N}_{V_i}(c) \cap W| \ge \frac{1}{2} \cdot \frac{1}{6} f^{-\tau} D^{\frac{1}{r-1}} \cdot |T_{W,2}| = \frac{1}{12} f^{-\tau} D^{\frac{1}{r-1}} |T_{W,2}|.$$

Using Lemma 52, similarly to (178) we now see that, with probability $1 - n^{-\omega(1)}$, for all such W we have

$$\frac{1}{12} f^{-\tau} D^{\frac{1}{r-1}} |T_{W,2}| \le |V_i \cap (T_{W,2} \times W)| \le \max\bigl\{4\epsilon^{-1}(|T_{W,2}| + |W|),\ p \cdot |T_{W,2}| \cdot |W| \cdot n^{2\epsilon}\bigr\}. \tag{183}$$

eq:TW2lu

We now analyze this estimate similarly to (178)–(179) above. Noting that $|W| = \Theta\Bigl((\log n)^{1-\frac{1}{r-1}} D^{\frac{1}{r-1}}\Bigr)$ and $p = \Theta(D^{-\frac{1}{r-1}})$, the term on the left-most side of (183) is much greater than $p \cdot |T_{W,2}| \cdot |W| \cdot n^{2\epsilon}$, so only the first term in the maximum on the right-hand side matters. Since $f^{-\tau} D^{\frac{1}{r-1}} \gg 1$, then (183) implies that

$$|T_{W,2}| \le 96\epsilon^{-1} |W| f^{\tau} / D^{\frac{1}{r-1}} = O\Bigl((\log n)^{1-\frac{1}{r-1}} f^{\tau}\Bigr). \tag{184}$$

eq:TW2

We now claim that

$$\max_{c \in T_{W,2}} \Bigl|\Bigl\{d :\ \{c,d\} \in B^{(2)}_{\binom{W}{2}}(i)\Bigr\}\Bigr| \le f^{\tau} \log n, \tag{185}$$

eq:upperboun

which suffices to complete the proof of Lemma (i) for $\ell = 4$. Indeed, by combining the definition of $|T_{W,2}|$ and (184) with the estimate (185), we then obtain

$$|P^{(2)}_{\binom{W}{2}}(i)| \le |B^{(2)}_{\binom{W}{2}}(i)| \le |T_{W,2}| \cdot f^{\tau} \log n \le \Theta\Bigl((\log n)^{2-\frac{1}{r-1}} f^{2\tau}\Bigr) \ll \frac{1}{2} f^{-\tau} D^{\frac{1}{r-1}} \le \frac{1}{2} f^{-2\tau} D^{\frac{2}{r-1}}, \tag{186}$$

PW2

which establishes the remaining case $j = 2$ of both (175) and (176), as desired.

It thus remains to prove (185), which is based on a case-by-case analysis. To avoid clutter, for any $\{c,d\} \in B^{(2)}_{\binom{W}{2}}(i)$ we write $U_{c,d}$ be the collection of all vertices u that appear in the definition (174) of $B^{(2)}_{\binom{W}{2}}(i)$ with respect to $\{c,d\}$. Note that $U_{c,d} \subseteq W$, and that by definition of $B^{(2)}_{\binom{W}{2}}(i)$, we have

$$|U_{c,d}| \ge \frac{1}{6} f^{-\tau} D^{\frac{1}{r-1}}.$$

Fixing $c \in T_{W,2}$, we are going to find an upper bound on $|U_{c,d} \cap U_{c,d'}|$ for distinct d, d'. Note that for $\{c,d\}$ and $\{c,d'\}$ in $B^{(2)}_{\binom{W}{2}}(i)$ and $u \in U_{c,d} \cap U_{c,d'}$, there are two copies H with vertex-set $\{c,d,u,v\}$ and H' with vertex-set $\{c,d',u,v'\}$ of K_4 minus two edges sitting on $\{c,d,u\}$ and $\{c,d',u\}$ in V_i, respectively. Then the number of such extensions $H \cup H'$ on $\{c,d,d'\}$ is an upper bound on $|U_{c,d} \cap U_{c,d'}|$. There are two possible graphs G_1, G_2 that are isomorphic to $H \cup H'$ depending on whether $v = v'$ or not: G_1 has 5 vertices $\tilde{c}, \tilde{d}, \tilde{d}', \tilde{u}, \tilde{v}$ and 5 edges $\{\tilde{c},\tilde{u}\}$, $\{\tilde{c},\tilde{v}\}$, $\{\tilde{u},\tilde{v}\}$, $\{\tilde{d},\tilde{v}\}$, $\{\tilde{d}',\tilde{v}\}$; G_2 has 6 vertices $\tilde{c}, \tilde{d}, \tilde{d}', \tilde{u}, \tilde{v}, \tilde{v}'$ and 7 edges $\{\tilde{c},\tilde{u}\}$, $\{\tilde{c},\tilde{v}\}$, $\{\tilde{c},\tilde{v}'\}$, $\{\tilde{u},\tilde{v}\}$, $\{\tilde{u},\tilde{v}'\}$, $\{\tilde{d},\tilde{v}\}$, $\{\tilde{d}',\tilde{v}\}$. Let $A=\{\tilde{c},\tilde{d},\tilde{d}'\}$ and an injection $\phi: A\to[n]$ is defined as $\phi(\tilde{c})=c, \phi(\tilde{d})=d, \phi(\tilde{d}')=d'$. Then the number of desired extensions $H\cup H'$ is the right-hand side of the following, hence

$$|U_{c,d}\cap U_{c,d'}|\le N_{\phi,G_1}(V_i)+N_{\phi,G_2}(V_i),$$

where $N_{\phi,G_i}(V_i)$ is defined in Section 4.5.1. By Corollary 55, we have $N_{\phi,G_j}\le n^{28\epsilon}\max_{A\subseteq B\subseteq V_{G_j}} S_{B,G_j}$. With $p=n^{-2/5}$ in mind, it is routine to check $\max_{A\subseteq B\subseteq V_{G_1}} S_{B,G_1}=S_{A\cup\{\tilde{v}\},G_1}=np^2=n^{1/5}$ and $\max_{A\subseteq B\subseteq V_{G_2}} S_{B,G_2}=S_{A,G_2}=n^3p^7=n^{1/5}$. Therefore

$$|U_{c,d}\cap U_{c,d'}|\le n^{29\epsilon}n^{1/5}.$$

After these preparations, we are now ready to complete the proof. For a fixed $c\in T_{W,2}$, assume there are distinct $d_1,\dots,d_\kappa$ with $\{c,d_j\}\in B^{(2)}_{\binom{W}{2}}(i)$ for $j=1,\dots,\kappa$. Aiming at a contradiction, assume that $\kappa\ge f^\tau\log n$. For $j=\lfloor f^\tau\log n\rfloor$, in view of $|W|=\Theta\Big((\log n)^{1-\frac{1}{r-1}}D^{\frac{1}{r-1}}\Big)$ it then follows that

$$\begin{aligned}\Big|\bigcup_{1\le b\le j}U_{c,d_b}\Big| &\ge \sum_{1\le b\le j}|U_{c,d_b}|-\sum_{1\le a,b\le j:a\ne b}|U_{c,d_a}\cap U_{c,d_b}|\\ &\ge \frac{j}{6}f^{-\tau}D^{\frac{1}{r-1}}-\frac{j^2}{2}n^{29\epsilon}n^{1/5}=\Theta\big(D^{\frac{1}{r-1}}\log n\big)\gg|W|,\end{aligned}$$

which contradicts that $\bigcup_{1\le b\le j}U_{c,d_b}\subseteq W$ (as observed above). Hence $\kappa\le f^\tau\log n$, which establishes (185) and thus completes the proof of Lemma (i) for $\ell=4$. □

A.2.2 Complete bipartite graph $K_{a,a}$: proof of Lemma 49 (iii)

th:indep:Kab

Lemma 49 (iii) follows from the following more general smoothness result for $K_{a,b}$.

ooth:general

Lemma 61. *The complete bipartite graph $K_{a,b}$ with fixed $a,b\ge3$ has the smooth independence property if $ab-2(a+b)+3>0$, $a^2-3a+3>b$, and $b^2-3b+3>a$. For fixed $a,b\ge4$, a sufficient condition for these inequalities to hold is $a\ge\sqrt{b}+2$ and $b\ge\sqrt{a}+2$, which in particular holds when $a=b$.*

Proof of Lemma 61 via Lemma 56. For $H=K_{a,b}$, and for two edges $\{x,y\},\{u,v\}$ with $H^-:=H\backslash\{u,v\}$, and $\{x,y\}\subsetneq B\subsetneq v_{H^-}$, we may assume $|B|=z_1+z_2$ for some $1\le z_1\le a, 1\le z_2\le b$ and $3\le z_1+z_2\le a+b-1$. As the assumption of Lemma (iii) guarantees that the minimum degree of H is at least 3, by the conclusion of Lemma 56 it remains to show for $p=n^{-\frac{a+b-2}{ab-1}}$ that

$$S_{B,H^-}=n^{a+b-z_1-z_2}p^{ab-z_1z_2-1}=n^{a+b-z_1-z_2}n^{-\frac{a+b-2}{ab-1}(ab-z_1z_2-1)}<1$$

holds for all z_1,z_2 in the range described as above. It thus is enough to show that

$$(a+b-z_1-z_2)-\frac{a+b-2}{ab-1}(ab-z_1z_2-1)<0,$$

or equivalently

$$\begin{aligned}g(z_1,z_2):=&(a+b-2)(ab-z_1z_2-1)-(ab-1)(a+b-z_1-z_2)\\ =&ab(z_1+z_2)-(a+b)z_1z_2-2ab+2z_1z_2+2-(z_1+z_2)>0\end{aligned}$$

for all possible z_1,z_2. Note that

$$\frac{\partial g}{\partial z_1}=ab-(a+b)z_2+2z_2-1=-(a+b-2)z_2+(ab-1).$$

Therefore

$$\begin{cases}\frac{\partial g}{\partial z_1}\ge0 & \text{if } z_2\le\frac{ab-1}{a+b-2},\\ \frac{\partial g}{\partial z_1}\le0 & \text{if } z_2\ge\frac{ab-1}{a+b-2}.\end{cases}$$

This implies that if $z_2 \le \frac{ab-1}{a+b-2}$ is fixed, then $g(\cdot, z_2)$ is an increasing function with respect to z_1. Furthermore, if $z_2 \ge \frac{ab-1}{a+b-2}$ is fixed, then $g(\cdot, z_2)$ is an decreasing function with respect to z_1. It follows that

$$\begin{cases} \frac{\partial g}{\partial z_2} \ge 0 & \text{if } z_1 \le \frac{ab-1}{a+b-2}, \\ \frac{\partial g}{\partial z_2} \le 0 & \text{if } z_1 \ge \frac{ab-1}{a+b-2}. \end{cases}$$

Therefore if $z_2 \le \frac{ab-1}{a+b-2}$, then

$$g(z_1, z_2) \ge \min\{g(1, z_2), g(2, z_2)\} \ge \min\{g(1,2), g(2,1)\},$$

if $1 \le 2 \le \frac{ab-1}{a+b-2}$ (which is true by our assumption). And it is not hard to check $g(1,2) = g(2,1) = ab - 2(a+b) + 3 > 0$ by our assumption. If $z_2 \ge \frac{ab-1}{a+b-2}$, then

$$g(z_1, z_2) \ge \min\{g(a, z_2), g(a-1, z_2)\} \ge \min\{g(a, b-1), g(a-1, b)\},$$

if $a \ge a-1 \ge \frac{ab-1}{a+b-2}$ (which is true by our assumption). And it is not hard to check $g(a, b-1) = a^2 - 3a - b + 3 > 0$ and $g(a-1, b) = b^2 - 3b - a + 3 > 0$ by our assumption, completing the proof. □

A.2.3 Cube Q^k: proof of Lemma 49 (iv)

oth:indep:Qk

Proof of Lemma 49 (iv) *via Lemma 56.* As a well-known fact, for any subgraph G of Q^k, average degree of G is at most $\log_2 v_G$ (see Lemma 4.1 in [12], for example), therefore the number of edges in G is at most $\frac{v_G \log_2 v_G}{2}$, so that $e_G \le \lfloor \frac{v_G \log_2 v_G}{2} \rfloor$. Therefore, for $H = Q^k$ and any two edges $\{x, y\}$, $\{u, v\}$, setting $H^- := H \setminus \{u, v\}$ and $p = n^{-\frac{2^k - 2}{k \cdot 2^{k-1} - 1}}$, by the conclusion of Lemma 56 it remains to show that

$$S_{B,H^-} = n^{v_{H^-} - |B|} p^{e_{H^-} - e_{H^-[B]}} \le n^{2^k - |B|} p^{k \cdot 2^{k-1} - 1 - \left\lfloor \frac{|B| \log_2 |B|}{2} \right\rfloor} < 1$$

for all $\{x, y\} \subsetneq B \subsetneq V_H$ with $3 \le |B| =: z \le 2^k - 1$. It thus is enough to show that

$$(2^k - z) - \frac{2^k - 2}{k \cdot 2^{k-1} - 1} \Big(k \cdot 2^{k-1} - 1 - \Big\lfloor \frac{z \log_2 z}{2} \Big\rfloor \Big) < 0,$$

which is equivalent to

$$(2^k - z)(k \cdot 2^{k-1} - 1) < (2^k - 2) \Big(k \cdot 2^{k-1} - 1 - \Big\lfloor \frac{z \log_2 z}{2} \Big\rfloor \Big), \tag{187}$$ floorbound

for all $3 \le z \le 2^k - 1$. Note that

$$\begin{aligned} \lambda(z) :=& (2^k - 2)\big(k \cdot 2^{k-1} - 1 - z \log_2(z)/2\big) - (2^k - z)(k \cdot 2^{k-1} - 1) \\ =& zk2^{k-1} - z - 2^{k-1} z \log_2 z - 2k \cdot 2^{k-1} + 2 + z \log_2 z > 0 \end{aligned}$$

for all $3 \le z \le 2^{k-1}$ will imply that (187) holds for all $3 \le z \le 2^{k-1}$. We have

$$\lambda'(z) = -(2^{k-1} - 1) \log_2 z + \Big(k \cdot 2^{k-1} - 1 - 2^{k-1} \frac{1}{\ln 2} + \frac{1}{\ln 2} \Big).$$

For $3 \le z \le 2^{k-1}$, we thus se that λ attains its minimum at either 3 or 2^{k-1}. For $k \ge 5$, we have

$$\begin{aligned} \lambda(2^{k-1}) &= 2^{k-1} k \cdot 2^{k-1} - 2^{k-1} - 2^{k-1} 2^{k-1} \log_2(2^{k-1}) - 2k \cdot 2^{k-1} + 2 + 2^{k-1} \log_2(2^{k-1}) \\ &= 2^{k-1}(2^{k-1} - 2k) + 2^{k-1}(k-2) + 2 > 0, \end{aligned}$$

and

$$\lambda(3) = (k - 3\log_2 3) \cdot 2^{k-1} + (3 \log_2 3 - 1) > 0.$$

For $2^{k-1} + 1 \le z \le 2^k - 1$, to show that (187) holds, firstly we are going to show

$$z \log_2 z < zk - (2^k - z).$$

Set $t := 2^k - z$, then $1 \le t \le 2^{k-1} - 1$. Our goal is equivalent to showing that

$$(2^k - t)\log_2(2^k - t) = (2^k - t)\Big(\log_2\Big(\frac{2^k - t}{2^k}\Big) + k\Big) < (2^k - t)k - t,$$

or, equivalently, to

$$(2^k - t)\log_2\Big(1 - \frac{t}{2^k}\Big) < -t$$

for all $1 \le t \le 2^{k-1} - 1$. Set $g(t) := (2^k - t)\log_2\big(1 - \frac{t}{2^k}\big) + t$, then

$$g'(t) = -\log_2\Big(1 - \frac{t}{2^k}\Big) + (2^k - t)\frac{1}{(1 - \frac{t}{2^k})\ln 2}\Big(-\frac{1}{2^k}\Big) + 1 = -\log_2\big(1 - \frac{t}{2^k}\big) - \frac{1}{\ln 2} + 1,$$

is a monotone strictly increasing function when $0 \le t \le 2^{k-1}$, with $g'(0) = 1 - \frac{1}{\ln 2} < 0$ and $g'(2^{k-1}) = 2 - \frac{1}{\ln 2} > 0$. Therefore when $0 \le t \le 2^{k-1}$, as t increasing, g strictly decreasing first and then strictly increasing, so g attains maximum at $t = 0$ or 2^{k-1}. Note that $g(0) = g(2^{k-1}) = 0$. Therefore $\max_{1 \le t \le 2^{k-1}-1} g(t) < g(0) = g(2^{k-1}) = 0$. It follows that

$$z\log_2 z < zk - (2^k - z)$$

for $2^{k-1} + 1 \le z \le 2^k - 1$. Then for some $\epsilon_0 > 0$, we have

$$\Big\lfloor \frac{z\log_2 z}{2} \Big\rfloor \le \Big\lfloor \frac{zk - (2^k - z) - \epsilon_0}{2} \Big\rfloor.$$

Since k is odd, therefore $zk - (2^k - z)$ is even. So

$$\Big\lfloor \frac{z\log_2 z}{2} \Big\rfloor \le \Big\lfloor \frac{zk - (2^k - z) - \epsilon_0}{2} \Big\rfloor \le \frac{zk - (2^k - z) - 2}{2}.$$

Go back to (187), it is enough to show for $2^{k-1} + 1 \le z \le 2^k - 1$,

$$h(z) := (2^k - 2)\Big(k \cdot 2^{k-1} - 1 - \frac{zk - (2^k - z) - 2}{2}\Big) - (2^k - z)(k \cdot 2^{k-1} - 1) > 0.$$

Factoring h we infer that

$$h(z) = (2^k - z)(2^{k-1} - k),$$

which is strictly greater than 0 when $2^{k-1} + 1 \le z \le 2^k - 1$ and $k \ge 5$, completing the proof. □

pseudorandom

A.3 Pseudo-random vertex-distribution (Section 5)

In this appendix we complete the proof of Theorem 58, by giving the proof of the lower bound in (169) that was deferred from Section 5. This proof is conceptually similar to the upper bound proof in (169) given in Section 5, but here we need to use more involved events to guarantee that certain vertices are added (these events are rather technical and not very insightful, which is why they are deferred to this appendix).

Proof of lower bound in (169). Keeping the setup and notations from the proof of Theorem 58 in Section 5, for step ℓ, let $h(\ell)$ be the smallest m_p for $p = 1, \dots, \tilde{\kappa}$ satisfying $m_p > \ell$. For later reference, we call steps m_p 'inclusion steps'. In each step ℓ, we need to guarantee $\cup_{q=h(\ell)}^{\tilde{\kappa}} J_q$, the inclusion steps of whose vertices are later, stays in A_ℓ, which in turn means their intersection with $\Gamma_\ell \cup C_\ell \cup_{z=2}^{r-1} C_\ell^{(z)}$ must be empty. With this in mind we define

$$\mathcal{C}_\ell := \big\{(\cup_{q=h(\ell)}^{\tilde{\kappa}} J_q) \cap (\Gamma_\ell \cup C_\ell \cup_{2 \le z \le r-1} C_\ell^{(z)}) = \varnothing\big\}. \tag{188}$$

eq:def:cCone

Furthermore, in an inclusion step m_p, we want J_p to be included in Γ_{m_p}, and none of its vertices is removed, i.e., $J_p \cap V(\mathcal{D}_{m_p}) = \varnothing$. (And, of course, $\cup_{q=h(m_p)}^{\tilde{\kappa}} J_q$, the inclusion steps of whose vertices are later, should stay in A_{m_p}, as discussed earlier.) For J_p to be included in Γ_{m_p}, we define

$$\mathcal{R}_{m_p} := \big\{J_p \subseteq \Gamma_{m_p}\big\}.$$

Note that if all the edges e with $e \cap J_p \neq \varnothing$ satisfy

$$e \setminus J_p \not\subseteq V_{m_p} = V_{m_p-1} \cup \Gamma_{m_p}, \tag{189}$$

eq:reqdanger

then $\{J_p \cap V(\mathcal{D}_{m_p}) = \varnothing\}$ holds.

There are two types of edges in danger of violating (189). The first type of 'dangerous" edges e are those satisfying $e \cap J_p \neq \varnothing$, $(e \setminus J_p) \subseteq A_{m_p-1} \cup V_{m_p-1}$, and $(e \setminus J_p) \cap A_{m_p-1} \neq \varnothing$. If $(e \setminus J_p) \cap A_{m_p-1} \subseteq \Gamma_{m_p}$, then it violates (189). Therefore we define the 'safeguard' event for inclusion step m_p as

$$\mathcal{S}^{in}_{m_p} := \bigcap_{\substack{e:\, e \cap J_p \neq \varnothing, \\ (e \setminus J_p) \subseteq A_{m_p-1} \cup V_{m_p-1}, \\ (e \setminus J_p) \cap A_{m_p-1} \neq \varnothing}} \big\{ (e \setminus J_p) \cap A_{m_p-1} \not\subseteq \Gamma_{m_p} \big\} \tag{190}$$

eq:def:Safin

to prevent the first type of dangerous edges e violating (189).

The second type of dangerous edges e are those satisfying $e \cap J_p \neq \varnothing$ and $(e \setminus J_p) \subseteq V_{m_p-1}$. Note that $\cap_{1 \le \ell \le m_p-1} \mathcal{C}_\ell$ already prevents the occurrence of those second type of dangerous edges e with $|e \cap J_p| = 1$, since otherwise the single vertex in $e \cap J_p$ is in $\cup_{1 \le \ell \le m_p-1} C_\ell$, a contradiction. Hence to avoid the occurrence of the remaining second type of dangerous edges e, i.e., those with $|e \cap J_p| \ge 2$, we define a prevention safeguard event for each step ℓ as

$$\mathcal{S}_\ell = \bigcap_{\substack{e:\, |e \cap (\cup_{q=h(\ell)}^{\tilde{\kappa}} J_q)| \ge 2, \\ \left(e \setminus (\cup_{q=h(\ell)}^{\tilde{\kappa}} J_q)\right) \subseteq A_{\ell-1} \cup V_{\ell-1}, \\ \left(e \setminus (\cup_{q=h(\ell)}^{\tilde{\kappa}} J_q)\right) \cap A_{\ell-1} \neq \varnothing}} \Big\{ \big(e \setminus (\cup_{q=h(\ell)}^{\tilde{\kappa}} J_q)\big) \cap A_{\ell-1} \not\subseteq \Gamma_\ell \Big\}. \tag{191}$$

eq:def:Safel

Define

$$\mathcal{E}^* := \bigcap_{p=1}^{\tilde{\kappa}} (\mathcal{R}_{m_p} \cap \mathcal{S}^{in}_{m_p}) \bigcap_{\ell=1}^{m_{\tilde{\kappa}}-1} (\mathcal{C}_\ell \cap \mathcal{S}_\ell) \cap \mathfrak{X}_{\le i}, \tag{192}$$

def:Evstar

and let $\mathcal{E}^*_j$ be the events that the first j steps are compatible with the event $\mathcal{E}^*$. By the above analysis, the event $\mathcal{E}^*$ implies the event $\{\cup_{p=1}^{\tilde{\kappa}} J_p \subseteq I_i\} \cap \mathfrak{X}_{\le i}$ in (169).

Then we move on to bound the probability

$$\mathbb{P}(\mathcal{E}^*) = \prod_{1 \le \ell \le i} \mathbb{P}(\mathcal{E}^*_\ell \mid \mathcal{E}^*_{\ell-1})$$

from below. Since $\mathbb{P}(\neg \mathfrak{X}_i) \le N^{-\omega(1)}$, later on it can inductively be shown that

$$\mathbb{P}(\mathcal{E}^*_{\ell-1}) \ge N^{-\Theta(1)} \tag{193}$$

eq:lowerboun

and thus

$$\mathbb{P}(\neg \mathfrak{X}_\ell \mid \mathcal{E}^*_{\ell-1}) = N^{-\omega(1)} \tag{194}$$

eq:probbadev

for all $1 \le \ell \le i$, which is negligible compared to the probabilities bounds that follow.

We start with steps $1 \le \ell \le m_{\tilde{\kappa}}$ such that $\ell \neq m_p$ for all $1 \le p \le \tilde{\kappa}$. We rewrite the event $\mathcal{C}_\ell$ defined in (188) as

$$\mathcal{C}_\ell = \underbrace{\{(\cup_{q=h(\ell)}^{\tilde{\kappa}} J_q) \cap C_\ell = \varnothing\}}_{=:\mathcal{C}_{\ell,1}} \cap \underbrace{\{(\cup_{q=h(\ell)}^{\tilde{\kappa}} J_q) \cap \Gamma_\ell = \varnothing\}}_{=:\mathcal{C}_{\ell,2}} \cap \underbrace{\{(\cup_{q=h(\ell)}^{\tilde{\kappa}} J_q) \cap (\cup_{2 \le z \le r-1} C_\ell^{(z)}) = \varnothing\}}_{=:\mathcal{C}_{\ell,3}},$$

and infer

$$\begin{aligned} \mathbb{P}(\mathcal{E}^*_\ell \mid \mathcal{E}^*_{\ell-1}) &= \mathbb{P}(\cap_{j=1}^3 \mathcal{C}_{\ell,j} \cap \mathcal{S}_\ell \cap \mathfrak{X}_\ell \mid \mathcal{E}^*_{\ell-1}) \\ &\ge \mathbb{P}(\mathcal{C}_{\ell,1} \mid \mathcal{E}^*_{\ell-1}) - \mathbb{P}(\neg \mathcal{C}_{\ell,2} \mid \mathcal{E}^*_{\ell-1}) - \mathbb{P}(\neg \mathcal{C}_{\ell,3} \mid \mathcal{E}^*_{\ell-1}) - \mathbb{P}(\neg \mathcal{S}_\ell \mid \mathcal{E}^*_{\ell-1}) - \mathbb{P}(\neg \mathfrak{X}_\ell \mid \mathcal{E}^*_{\ell-1}). \end{aligned} \tag{195}$$

eq:lowerprob

In the following we shall estimate all the terms in (195), where we think of $\mathbb{P}(\mathcal{R}_\ell \mid \mathcal{E}^*_{\ell-1})$ as the main term. Note that, by Lemma 60, we have

$$\mathbb{P}(\mathcal{C}_{\ell,1} \mid \mathcal{E}^*_{\ell-1}) \geq (1-\sigma^{1+\rho/2}\pi_{\ell-1}^{\max\{r-3,0\}})\Big(\frac{q_\ell}{q_{\ell-1}}\Big)^{\sum_{q=h(\ell)}^{\tilde\kappa}|J_q|}. \tag{196}$$

eq:error1cl1

Using the fact that $|\cup_{q=h(\ell)}^{\tilde\kappa} J_q)| \leq \kappa$, we also infer that

$$\mathbb{P}(\neg\mathcal{C}_{\ell,2} \mid \mathcal{E}^*_{\ell-1}) \leq \kappa p_{\ell-1} \tag{197}$$

eq:errorcl2

and

$$\mathbb{P}(\neg\mathcal{C}_{\ell,3} \mid \mathcal{E}^*_{\ell-1}) \leq \sum_{z=2}^{r-1} \kappa \cdot \max_{v\in A_0}|Y_{v,z}(\ell-1)| \cdot p_{\ell-1}^z. \tag{198}$$

eq:errorcl3

For any edge e, we have $|e \cap (\cup_{q=h(\ell)}^{\tilde\kappa} J_q)| \leq r-1$ by the fact that $\{v_1,\dots,v_\kappa\}$ contains no edge of $\mathcal{H}$. Once we know the sizes $a = |e \cap (\cup_{q=h(\ell)}^{\tilde\kappa} J_q)|$ and $j = |\big(e \setminus (\cup_{q=h(\ell)}^{\tilde\kappa} J_q)\big) \cap A_{\ell-1}|$, the number of the edges e in the definition (191) of $\mathcal{S}_\ell$ is at most $\binom{\sum_{q=h(\ell)}^{\tilde\kappa}|J_q|}{a} \cdot \Delta_{a,r-a-j}(\ell-1)$. Therefore

$$\mathbb{P}(\neg\mathcal{S}_\ell \mid \mathcal{E}^*_{\ell-1}) \leq \sum_{a=2}^{\min\{r-1,\ \sum_{q=h(\ell)}^{\tilde\kappa}|J_q|\}} \sum_{j=1}^{r-a} \binom{\sum_{q=h(\ell)}^{\tilde\kappa}|J_q|}{a} \cdot \Delta_{a,r-a-j}(\ell-1)\cdot p_{\ell-1}^j. \tag{199}$$

eq:Sellneqmp

To estimate (198), using $\mathfrak{X}_{\ell-1} \subseteq \mathcal{Y}_{\ell-1}$, $\pi_j \geq \sigma$, and $\rho < 1$, for $2 \leq z \leq r-1$ (which enforces $r \geq 3$) we have

$$\begin{aligned} p_{\ell-1}^z \max_{v\in A_0}|Y_{v,z}(\ell-1)| &\leq \Big(\frac{\sigma}{q_{\ell-1}D^{\frac{1}{r-1}}}\Big)^z \binom{r-1}{z} q_{\ell-1}^z \pi_{\ell-1}^{r-1-z} D^{\frac{z}{r-1}} \\ &\leq \binom{r-1}{z}\sigma^{1+\rho/2}\pi_{\ell-1}^{\max\{r-3,0\}}\sigma^{1-\rho/2} \\ &\ll \sigma^{1+\rho/2}\pi_{\ell-1}^{\max\{r-3,0\}}. \end{aligned}$$

To estimate (197) and (199), using $\mathfrak{X}_{\ell-1} \subseteq \mathcal{N}_{\ell-1}$, (20), and (40) from Lemma 18, for $2 \leq a \leq r-1$, $1 \leq j \leq r-a$ similarly we have

$$\max\{\kappa p_{\ell-1},\ \ \Delta_{a,r-a-j}(\ell-1)\cdot p_{\ell-1}^j\} \ll \sigma^{1+\rho/2}\pi_{\ell-1}^{\max\{r-3,0\}}.$$

Putting (197), (198), and (199) together, we obtain

$$\mathbb{P}(\neg\mathcal{C}_{\ell,2} \mid \mathcal{E}^*_{\ell-1}) + \mathbb{P}(\neg\mathcal{C}_{\ell,3} \mid \mathcal{E}^*_{\ell-1}) + \mathbb{P}(\neg\mathcal{S}_\ell \mid \mathcal{E}^*_{\ell-1}) = o(\sigma^{1+\rho/2}\pi_{\ell-1}^{\max\{r-3,0\}}). \tag{200}$$

eq:cl2cl3s

Combining (195), (196), and (200), and using the fact that $\frac{q_\ell}{q_{\ell-1}} \sim 1$ by Remark 28, we obtain

$$\mathbb{P}(\mathcal{E}^*_\ell \mid \mathcal{E}^*_{\ell-1}) \geq (1-2\sigma^{1+\rho/2}\pi_{\ell-1}^{\max\{r-3,0\}})\Big(\frac{q_\ell}{q_{\ell-1}}\Big)^{\sum_{q=h(\ell)}^{\tilde\kappa}|J_q|}. \tag{201}$$

eq:lowerprob

Next we turn to inclusion step $\ell = m_p$ for some $1 \leq p \leq \tilde\kappa$. We have

$$\begin{aligned} &\mathbb{P}(\mathcal{E}^*_\ell \mid \mathcal{E}^*_{\ell-1}) \\ =&\mathbb{P}(\mathcal{R}_\ell \cap \mathcal{C}_\ell \cap \mathcal{S}^{in}_\ell \cap \mathcal{S}_\ell \cap \mathfrak{X}_\ell \mid \mathcal{E}^*_{\ell-1}) \\ \geq&\mathbb{P}(\mathcal{R}_\ell \mid \mathcal{E}^*_{\ell-1})\mathbb{P}(\mathcal{C}_\ell \cap \mathcal{S}^{in}_\ell \cap \mathcal{S}_\ell \mid \mathcal{R}_\ell \cap \mathcal{E}^*_{\ell-1}) - \mathbb{P}(\neg\mathfrak{X}_\ell \mid \mathcal{E}^*_{\ell-1}) \\ \geq&\mathbb{P}(\mathcal{R}_\ell \mid \mathcal{E}^*_{\ell-1})\Big(1-\mathbb{P}(\neg\mathcal{C}_\ell \mid \mathcal{R}_\ell \cap \mathcal{E}^*_{\ell-1}) - \mathbb{P}(\neg\mathcal{S}^{in}_\ell \mid \mathcal{R}_\ell \cap \mathcal{E}^*_{\ell-1}) - \mathbb{P}(\neg\mathcal{S}_\ell \mid \mathcal{R}_\ell \cap \mathcal{E}^*_{\ell-1})\Big) - \mathbb{P}(\neg\mathfrak{X}_\ell \mid \mathcal{E}^*_{\ell-1}). \end{aligned} \tag{202}$$

eq:prodofpro

Note that

$$\mathbb{P}(\mathcal{R}_\ell \mid \mathcal{E}^*_{\ell-1}) = p_{\ell-1}^{|J_p|}. \tag{203}$$

eq:prodofpro

In the following we shall analyze the remaining terms in (202) term by term, where we think of $\mathbb{P}(\mathcal{R}_\ell \mid \mathcal{E}^*_{\ell-1})$ as the main term. We rewrite $\mathcal{C}_\ell$ defined in (188) as

$$\mathcal{C}_\ell = \underbrace{\{\cup_{q=h(\ell)}^{\tilde\kappa} J_q \cap \Gamma_\ell = \varnothing\}}_{:=\mathcal{C}^{in}_{\ell,1}} \bigcap \underbrace{\{\cup_{q=h(\ell)}^{\tilde\kappa} J_q \cap S_\ell = \varnothing\}}_{:=\mathcal{C}^{in}_{\ell,2}} \bigcap \underbrace{\cap_{u \in \cup_{q=h(\ell)}^{\tilde\kappa} J_q} \cap_{z=1}^{r-1} \cap_{W \in Y_{u,z}(\ell-1)} \{W \not\subseteq \Gamma_\ell\}}_{:=\mathcal{C}^{in}_{\ell,3}}. \tag{204}$$

eq:rewritecC

We have

$$\mathbb{P}(\neg \mathcal{C}^{in}_{\ell,1} \mid \mathcal{R}_\ell \cap \mathcal{E}^*_{\ell-1}) \le \kappa p_{\ell-1} = o((\log f)^{-2}). \tag{205}$$

eq:rewriteCe

By (24), (20), (39), and Bernoulli's inequality $(1-x)^y \ge 1 - xy$ for $x \in (0,1)$ and $y \in \{0\} \cup [1,\infty)$, we have

$$\mathbb{P}(\neg \mathcal{C}^{in}_{\ell,2} \mid \mathcal{R}_\ell \cap \mathcal{E}^*_{\ell-1}) \le \kappa \max_{v \in A_{\ell-1}} \hat p_{v,\ell-1} \le \kappa \cdot p_i (r-1) q_i \big(\pi_i^{r-2} + \sigma^\rho \pi_i^{\max\{r-3,0\}}\big) D^{\frac{1}{r-1}} = o((\log f)^{-2}). \tag{206}$$

eq:rewriteCe

For $\mathcal{C}^{in}_{\ell,3}$, we distinguish the size of $a = |W \cap J_p|$ for $W \in Y_{u,z}(\ell-1)$ for some $u \in \cup_{q=p+1}^{\tilde\kappa} J_q$ and $1 \le z \le r-1$. We have $0 \le a < r-1$, since $\cup_{q=1}^{\tilde\kappa} J_q$ contains no edge of $\mathcal{H}$. If $a = 0$, then their contribution is at most the first part of (207). If $1 \le a \le \min\{r-2,\ |J_p|\}$, the safeguard events $\cap_{1 \le t \le m_p - 1} \mathcal{S}_t$ ensure that $W \setminus J_p \neq \varnothing$: otherwise let e be an edge such that $u \in e$, $W = e \cap A_{\ell-1} \subseteq J_p$, and $e \setminus (\{u\} \cup W) \subseteq V_{\ell-1}$ by the definition of $W \in Y_{u,z}(\ell-1)$, but the occurrence of such e is prevented by $\cap_{1 \le t \le m_p-1} \mathcal{S}_\ell$ since $|e \cap (\cup_{q=p}^{\tilde\kappa} J_q)| \ge 2$. Setting $1 \le j = |W \setminus J_p| \le r - (a+1)$ and using $W \setminus J_p \subseteq A_{\ell-1}$, their contribution is at most the second part of (207). Summing up, we have

$$\begin{aligned}
&\mathbb{P}(\neg \mathcal{C}^{in}_{\ell,3} \mid \mathcal{R}_\ell \cap \mathcal{E}^*_{\ell-1}) \\
\le& \sum_{q=p+1}^{\tilde\kappa} |J_q| \Big(\sum_{z=1}^{r-1} \max_{u \in A_0} |Y_{u,z}(\ell-1)| \cdot p_{\ell-1}^z + \sum_{a=1}^{\min\{r-2,\ |J_p|\}} \sum_{j=1}^{r-a-1} \binom{|J_p|}{a} \Delta_{a+1, r-a-1-j}(\ell-1) \cdot p_{\ell-1}^j \Big) \\
=& o((\log f)^{-2}).
\end{aligned} \tag{207}$$

eq:rewriteCe

Combining (204) with (205)– (207), we have

$$\mathbb{P}(\neg \mathcal{C}_\ell \mid \mathcal{R}_\ell \cap \mathcal{E}^*_{\ell-1}) = o((\log f)^{-2}). \tag{208}$$

eq:probrewri

By distinguishing the size of $1 \le a = |e \cap J_p| \le \min\{r-1, |J_p|\}$ and $j = |(e \setminus J_p) \cap A_{m_p-1}|$ to bound the number of edges e in the definition of $\mathcal{S}^{in}_\ell$, we infer

$$\begin{aligned}
&\mathbb{P}(\neg \mathcal{S}^{in}_\ell \mid \mathcal{R}_\ell \cap \mathcal{E}^*_{\ell-1}) \\
\le& \Big(|J_p| \sum_{j=1}^{r-1} \max_{v \in A_0} |Y_{v,j}(\ell-1)| \cdot p_{\ell-1}^j + \sum_{a=2}^{\min\{r-1,\ |J_p|\}} \sum_{j=1}^{r-a} \binom{|J_p|}{a} \Delta_{a, r-a-j}(\ell-1) \cdot p_{\ell-1}^j \Big) = o((\log f)^{-2}).
\end{aligned} \tag{209}$$

eq:prodofpro

Similar as (199), we obtain

$$\mathbb{P}(\neg \mathcal{S}_\ell \mid \mathcal{R}_\ell \cap \mathcal{E}^*_{\ell-1}) \le \sum_{a=2}^{\min\{r-1,\ \sum_{q=p+1}^{\tilde\kappa} |J_q|\}} \sum_{j=1}^{r-a} \binom{\sum_{q=p+1}^{\tilde\kappa} |J_q|}{a} \cdot \Delta_{a, r-a-j}(\ell-1) \cdot p_{\ell-1}^j = o((\log f)^{-2}). \tag{210}$$

eq:prodofpro

By (202), combining (208), (209), and (210) with (203) and (194), we have

$$\mathbb{P}(\mathcal{E}^*_\ell \mid \mathcal{E}^*_{\ell-1}) \ge (1 - o(\log f)^{-2}) \Big(\frac{\sigma}{q_{m_p-1} D^{\frac{1}{r-1}}} \Big)^{|J_p|}. \tag{211}$$

eq:prodofpro

Next we turn to the steps satisfying $m_{\tilde\kappa} < \ell \le i$, which requires one extra technical estimate. Namely, multiplying (201) and (211) up to step $m_{\tilde\kappa}$, using $m\sigma^{1+\rho/2} \pi_i^{\max\{r-3,0\}} \ll 1$ (see (15) together with (39) from Lemma 18) and $q_{t-1}/q_t \le 1 + O(\sigma^{1/2})$ (see Remark 28), verifies (193) for $\ell - 1 = m_{\tilde\kappa}$. Closely inspecting the event $\mathcal{E}^*_{\ell-1}$ that the first ℓ steps are compatible with the event $\mathcal{E}^*$ defined in (192), using (194) it thus inductively follows that, for $m_{\tilde{\kappa}} < \ell \le i$,

$$\mathbb{P}(\mathcal{E}^*_\ell \mid \mathcal{E}^*_{\ell-1}) = 1 - \mathbb{P}(\neg\mathfrak{X}_\ell \mid \mathcal{E}^*_{\ell-1}) \ge 1 - N^{-\omega(1)}. \tag{212}$$

eq:pseudoran

Finally, by multiplying (201), (211), and (212) over all steps, using $m\sigma^{1+\rho/2}\pi_i^{\max\{r-3,0\}} \ll 1$ (see (15) and (39) in Lemma 18) and similar cancellation as the computation in the upper bound, it follows that

$$\mathbb{P}(\mathcal{E}^*) \ge (1+o(1))\Big(\frac{\sigma}{D^{\frac{1}{r-1}}}\Big)^{\kappa},$$

completing the proof of the lower bound in (169) by multiplying with all i^κ possibilities (similar as we did in Section 5, for the proof of the corresponding upper bound). $\square$