TOPOLOGICAL HALL

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ABSTRACT. A topological version of Hall's marriage theorem, essentially proved in [13], provides a sufficient condition for the existence of a choice function from a given family of sets, whose image belongs to a given complex (closed down hypergraph). A homological version was proved by Meshulam, while the homotopic version, which came first, has mostly remained folklore. We formulate and prove the homotopic version (with which combinatorialists may feel more comfortable), and prove an injective version. For this purpose we develop homotopic tools, parallel to those provided by homology. We then survey some of the main applications and open problems.

1. INTRODUCTION

1.1. **Why is topology relevant to combinatorics?** Since Lovàsz's seminal solution of Kneser's conjecture [33], topology has become a basic tool in combinatorics. There is an ever-growing list of results for which the only known proofs are topological.

The secret is in that topology has a say on the intersection patterns of sets. The most basic such fact is that two non-empty closed subsets of $[0, 1]$ covering the interval must intersect. This is the 1-dimensional case of the famous Knaster–Kuratowski–Mazurkiewicz (KKM) theorem [32], which we now describe.

Let $\Delta_{[m]}$ be the standard $(m-1)$-dimensional simplex, i.e.,

$$\Delta_{[m]} := \text{conv}(\{e_1, \dots, e_m\}) = \Big\{ \sum_{i=1}^{m} \lambda_i e_i \mid \lambda_i \geq 0 \text{ for } 1 \leq i \leq m \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \Big\}.$$

(Here, as usual, e_i is the i-th standard basis vector, $e_i(j) = \delta^i_j$.) For $I \subseteq [m]$, let

$$\Delta_I := \text{conv}(\{e_i \mid i \in I\}). \tag{1}$$

Theorem 1.1 (KKM)**.** *Let $C_1, \dots, C_m$ be closed subsets of $\Delta_{[m]}$. If $\Delta_I \subseteq \cup_{i \in I} C_i$ for every $I \subseteq [m]$, then*

$$\cap_{i \in [m]} C_i \neq \varnothing.$$

Independently, at around the same time that this theorem was proved, Emanuel Sperner [41] found a discrete version, easily derivable from it and easily implying it. A *Sperner coloring* for a triangulation $\mathcal{T}$ of $\Delta_{[m]}$ is a map $\lambda : V(\mathcal{T}) \to [m]$ satisfying $\lambda(v) \in \{i \in [m] \mid v \cdot e_i \neq 0\}$ for any $v \in V(\mathcal{T})$. (A triangulation is a partition into simplices — the notion is made precise in Definition 2.1 below.)

Lemma 1.2 (Sperner)**.** *For any triangulation $\mathcal{T}$ of $\Delta_{[n]}$ and a Sperner coloring for $\mathcal{T}$, there exist oddly many $(n-1)$-dimensional simplices in $\mathcal{T}$, whose n vertices receive n distinct colors.*

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Sperner used this lemma to give a simple proof of Brouwer's fixed point theorem which is stated in the following.

Theorem 1.3 (Brouwer)**.** *Any continuous function from a closed convex set in $\mathbb{R}^n$ to itself has a fixed point.*

It is also easy to derive Sperner's lemma from Brouwer's theorem, and thus the three results - KKM, Brouwer and Sperner - form a triad of equivalent statements (in the sense of two-way easy implications). Lovász's theorem belongs to another family - it uses Borsuk's theorem, which is one level higher, in the sense that it easily implies Brouwer, but not vice versa. The theorem discussed in this paper, a topological version of Hall's marriage theorem, belongs to the Brouwer family. It is easily derivable from any of the three members of the triad, but no proof is known in the other direction. There are many other results belonging to this family, see [34].

Since its inception in 2000, the topological version of Hall's theorem has had diverse applications, in many cases presently obtainable only through it. In this paper, we describe its proof and some tools that, once added to it, yield combinatorial insights.

2. Complexes

The tool bridging between topology and combinatorics is that of *simplicial complexes*. For completeness, we define the relevant terminology below. Readers familiar with these notions can skip this section.

2.1. **Abstract simplicial complexes.** An *(abstract) simplicial complex* $\mathcal{C}$ on ground set V is a finite collection of finite subsets of V that is closed-down, i.e., if $\sigma \in \mathcal{C}$, then $\tau \in \mathcal{C}$ for any $\tau \subseteq \sigma$. In this paper, we always think of $\emptyset$ as a member of every simplicial complex, so by the *empty complex*, we mean $\{\emptyset\}$ rather than $\emptyset$.

The members of a simplicial complex (i.e. the edges of the hypergraph) are called *faces*, or *simplices*, the last term derived from the geometric realization, to be described below. The elements of $V(\mathcal{C})$ are called a *points*, to differentiate them from *vertices* of graphs we shall meet. We shall assume that $\{v\} \in \mathcal{C}$ for every $v \in V(\mathcal{C})$ (so there are no idle points). For $U \subseteq V(\mathcal{C})$, we denote by $\mathcal{C}[U]$ the subcomplex of $\mathcal{C}$ induced by U, namely $\{e \in \mathcal{C} \mid e \subseteq U\}$. The *rank* $\mathrm{rank}(\mathcal{C})$ of a complex $\mathcal{C}$ is the maximal size of a face, and the *dimension* $\dim(\mathcal{C})$ of $\mathcal{C}$ is $\mathrm{rank}(\mathcal{C}) - 1$.

2.2. **Geometric simplicial complexes.** A set A of points in $\mathbb{R}^d$ is called *affinely independent* if $\{(1, x) \mid x \in A\}$ is linearly independent. A *geometric simplex* σ is the convex hull of an affinely independent set A. The points of the set A are called *extreme points* of the simplex. Then the convex hull of an arbitrary subset of A is called a *face* of σ. The *dimension* $\dim(\sigma)$ of the simplex σ thus defined is $|A| - 1$. A non-empty finite family $\mathcal{D}$ of geometric simplices is called a *geometric simplicial complex* if (i) each face of any geometric simplex $\sigma \in \mathcal{D}$ is also a geometric simplex in $\mathcal{D}$, and (ii) the intersection $\sigma_1 \cap \sigma_2$ of any two geometric simplices $\sigma_1, \sigma_2 \in \mathcal{D}$ is a face of both σ_1 and σ_2.

For a geometric simplicial complex $\mathcal{D}$, we write $V(\mathcal{D})$ for the set of extreme points of simplices in $\mathcal{D}$, and write $||\mathcal{D}||$ for $\bigcup \mathcal{D}$ and call it the *polyhedron* of $\mathcal{D}$. The *face poset* of a geometric simplicial complex $\mathcal{D}$ is the set of faces of simplices of $\mathcal{D}$ ordered by inclusion.

Definition 2.1. If a subset S of $\mathbb{R}^n$ is homeomorphic to $||\mathcal{D}||$, we say that $\mathcal{D}$ is a triangulation of S.

A geometric simplicial complex $\mathcal{D}_1$ is a *subdivision* of another geometric simplicial complex $\mathcal{D}_2$ if $||\mathcal{D}_1|| = ||\mathcal{D}_2||$ and every simplex in $\mathcal{D}_1$ is a subset of some simplex in $\mathcal{D}_2$.

A geometric complex $\mathcal{D}$ is called a *PL d-ball* (or $(d-1)$-sphere) if it admits a subdivision whose face poset is isomorphic to the face poset of some subdivision of $\Delta_{[d+1]}$ (respectively, the boundary of $\Delta_{[d+1]}$). (PL stands for piecewise linear.)

2.3. **Geometric realization of abstract complexes.** The collection of sets of extreme points of all faces of a geometric complex is an abstract complex. Conversely, for an abstract simplicial complex $\mathcal{C}$ with $|V(\mathcal{C})|-1 = n$, if we identify $V(\mathcal{C})$ with the set of extreme points of the n-dimensional simplex, then

$$\mathcal{D} := \{\text{conv}(\sigma) \mid \sigma \in \mathcal{C}\}$$

is a geometric simplicial complex.

In general, for an abstract simplicial complex $\mathcal{C}$, if an injective map $f : V(\mathcal{C}) \to \mathbb{R}^n$ satisfying that

$$\mathcal{D} := \{\text{conv}(f[\sigma]) \mid \sigma \in \mathcal{C}\}$$

is a geometric simplicial complex, then $\mathcal{D}$ is called a *geometric realization* of $\mathcal{C}$. (Here and elsewhere $f[S]$ denotes the image of the set S under the function f.)

Example 2.2. A geometric realization of $2^{[m]}$ (the power set of $[m]$) is $\Delta_{[m-1]}$, and a geometric realization of $2^{[m]} \setminus \{[m]\}$ is the boundary of $\Delta_{[m-1]}$.

It is easy to prove that two polyhedra of any realizations of a simplicial complex $\mathcal{C}$ (even in Euclidean spaces of different dimensions) are homeomorphic (see, e.g., [21, Proposition B.42]). The following is a sharp result concerning the dimension of the Euclidean space that a d-dimensional abstract complex can be geometrically realized. (See, e.g., [34, Theorem 1.6.1].)

Theorem 2.3 (Geometric realization theorem)**.** *An abstract complex of dimension d can be realized geometrically in $\mathbb{R}^{2d+1}$.*

Example 2.4. The 1-dimensional case is the familiar fact that graphs can be realized in $\mathbb{R}^3$, in such a way that edges meet only in original vertices of the graph. And $\mathbb{R}^3$ is the best possible because there are non-planer graphs.

The $1-1$ correspondence between abstract and geometric complexes makes the distinction between the two redundant, "complex" will mean here both.

2.4. **Some operations on complexes.** For two complexes $\mathcal{C}_1$ and $\mathcal{C}_2$, a map $\psi : V(\mathcal{C}_1) \to V(\mathcal{C}_2)$ is called *simplicial* if $\psi[\sigma] \in \mathcal{C}_2$ for every simplex σ in $\mathcal{C}_1$.

2.4.1. *Links.*

Definition 2.5. The *link* of a face σ of a complex $\mathcal{C}$, denoted $\text{lk}_{\mathcal{C}}(\sigma)$, is the complex $\{\tau \mid \sigma\cap\tau = \emptyset,\ \sigma\cup\tau \in \mathcal{C}\}$. For a point v, we abbreviate and write $\text{lk}_{\mathcal{C}}(v)$ for $\text{lk}_{\mathcal{C}}(\{v\})$.

This is also sometimes (especially in the case of matroids) called the *quotient* of $\mathcal{C}$ by σ, and is denoted by $\mathcal{C}/\sigma$

Example 2.6. $lk_{2^{[m]}}(m) = 2^{[m-1]}$.

Theorem 2.7 (e.g., Section 11 in [18])**.** *The link of a point in a PL m-dimensional sphere is a PL $(m-1)$-dimensional sphere.*

2.4.2. *Joins.*

Definition 2.8. Given two abstract complexes $\mathcal{C}$ and $\mathcal{D}$ on disjoint ground sets, their *join* $\mathcal{C} * \mathcal{D}$ is

$$\{\sigma \cup \tau : \sigma \in \mathcal{C}, \tau \in \mathcal{D}\}.$$

Geometrically, $||\mathcal{C} * \mathcal{D}||$ can be obtained by embedding $||\mathcal{C}||$ and $||\mathcal{D}||$ in perpendicular spaces and defining

$$||\mathcal{C} * \mathcal{D}|| = \{tx + (1-t)y : t \in [0,1],\, x \in ||\mathcal{C}||,\, y \in ||\mathcal{D}||\},$$

and we note that since the spaces are perpendicular, for every point in $||\mathcal{C} * \mathcal{D}|| \setminus (||\mathcal{C}|| \cup ||\mathcal{D}||)$, the choice of (t, x, y) is unique (see, e.g., [34, Proposition 4.2.4]).

When $\mathcal{C} = \{\varnothing, \{x\}\}$, we abbreviate $\mathcal{C} * \mathcal{D}$ as $x * \mathcal{D}$.

If the ground sets are not disjoint, we replace $\mathcal{D}$ by its copy on a ground set disjoint from $V(\mathcal{C})$. Clearly, the join operation is commutative and associative, so the join of more than two complexes is unambiguously defined.

We recall that by the "empty complex" we mean $\{\emptyset\}$ rather than $\emptyset$. This implies that if $\mathcal{C}$ is the empty complex and $\mathcal{D}$ is any complex then $\mathcal{C} * \mathcal{D} = \mathcal{D}$.

3. Connectivity

The central topological notion we shall use is that of *connectivity*, of a topological space in general and of complexes in particular. Intuitively, the connectivity of a complex is the smallest dimension of a hole in the complex, the hole being the missing part, not the boundary, which we assume to be contained in the complex. Rigorously, the following is the definition.

Definition 3.1. A topological space X is *(homotopically) k-connected* if for every $-1 \leq i \leq k$, every continuous function from S^i to $\mathcal{C}$ can be extended to a continuous function from B^{i+1} to $\mathcal{C}$. A complex $\mathcal{C}$ is *(homotopical) k-connected* if its geometric realization is k-connected.

Equivalently, a topological space X is k-connected if for every $-1 \leq i \leq k$, the ith homotopy group $\pi_i(X)$ is trivial.

Definition 3.2. A complex $\mathcal{C}$ is said to be *(simplicially) k-connected* if for every $-1 \leq i \leq k$, for every triangulation $\mathcal{T}$ of S^i and every simplicial map $\psi : \mathcal{T} \to \mathcal{C}$, there exist a triangulation $\mathcal{T}'$ of B^{i+1} and a simplicial map $\psi' : \mathcal{T}' \to \mathcal{C}$ such that $\mathcal{T}' \restriction S^i = \mathcal{T}$ and $\psi' |_{\mathcal{T}} = \psi$.

The fact that the simplicial k-connectivity of $\mathcal{C}$ and homotopical k-connectivity of $\mathcal{C}$ are equivalent is folklore.[1] Since the two are equivalent, we shall omit the "simplicial" prefix, and speak only of "k-connectivity".

Definition 3.3. The *(homotopical) connectivity* $\eta(\mathcal{C})$ is the maximal k such that $\mathcal{C}$ is k-connected, plus 2. We write $\eta(\mathcal{C}) = \infty$ if $\mathcal{C}$ is k-connected for every k.

The definition of $\eta(X)$ for a topological space X is the same.

By definition, $\eta(\mathcal{C}) \geq 0$ for all $\mathcal{C}$. S^{-1} is defined as $\varnothing$ and B^0 as a single point, so $\eta(\mathcal{C}) = 0$ means the absence of singleton faces. $\eta \geq 2$ means that every copy of S^0 (namely every pair of two distinct points) is fillable by a triangulation of B^1,

[1] If $\mathcal{C}$ is simplicially k-connected, then by simplicial approximation theorem (see, e.g., [26, Section 2.C]), it is homotopically k-connected. If $\mathcal{C}$ is homotopically k-connected, then by a relative version of simplicial approximation [44], it is simplicially k-connected.

namely by a path, so $\eta > 1$ is tantamount to path-connectedness. $\eta \geq 3$ means simple connectedness. $\eta \geq k$ means that holes can be filled whenever the filling demands simplices of size k or less.

Example 3.4 (e.g., Theorem 4.3.2 in [34]). $\eta(S^k) = k+1$, $\eta(B^k) = \infty$.

Example 3.5. For a complex $\mathcal{C}$ and a point v, $\eta(\mathcal{C} * v) = \infty$. It is because for any triangulation $\mathcal{T}$ of S^i and simplicial map $\psi : \mathcal{T} \to \mathcal{C} * v$, take a point $x \notin V(\mathcal{T})$, set $\mathcal{T}' = \mathcal{T} * x$ and $\psi' \mid_{V(\mathcal{T})} = \psi$ and $\psi'(x) = v$. Then $\mathcal{T}'$ is a triangulation of B^{i+1} extending $\mathcal{T}$ and ψ' is a simplicial map extending ψ.

Example 3.6. The d-dimensional *crosspolytope* $\mathrm{conv}(\{e_1, -e_1, \ldots, e_d, -e_d\})$ is the convex hull of d standard orthonormal basis vectors and their negatives. Note that it is homeomorphic to B^d. A triangulation of its boundary has $\sigma \subseteq V$ such that $|\{e_i, -e_i\} \cap \sigma| \leq 1$ for each $1 \leq i \leq d$, as faces. We denote this complex by X^d. Then X^d is a PL S^{d-1}, so $\eta(X^d) = d$. This complex is a partition matroid, the parts being $\{e_i, -e_i\}_{1 \leq i \leq d}$. Its rank (the number of parts) is d.

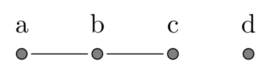

FIGURE 1. An abstract complex $\mathcal{C}$ with $\eta = 1$.
The sub-complex $\mathcal{C}[\{a, c\}]$ is a 0-sphere (the boundary of a 1-simplex); it is contained in the sub-complex $\mathcal{C}[\{a, b, c\}]$, which is a triangulation of a 1-ball (the same 1-simplex).
The sub-complex $\mathcal{C}[\{c, d\}]$ is a 0-sphere that is not contained in a 1-ball; it is a 1-dimensional hole.

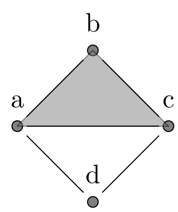

FIGURE 2. An abstract complex $\mathcal{C}$ with $\eta = 2$.
The sub-complex $\mathcal{C}[\{a, b\}] \cup \mathcal{C}[\{a, c\}] \cup \mathcal{C}[\{c, a\}]$ is a 1-sphere (the boundary of a 2-simplex); it is contained in the sub-complex $\mathcal{C}[\{a, b, c\}]$, which is a 2-ball (2-simplex).
However, the sub-complex $\mathcal{C}[\{a, c, d\}]$ is a 1-sphere that is not contained in a 2-ball; there is a 2-dimensional "hole".

Remark 3.7. Our terminology deviates from the standard one, in which "connectivity" is without the "plus 2". The present definition simplifies the formulation of Topological Hall (Theorem 4.3), and of other results.

Let

$$\bar{\eta}(\mathcal{C}) := \min(\eta(\mathcal{C}), \text{rank}(\mathcal{C})). \tag{2}$$

Claim 3.8. *$\eta(\mathcal{C}) = \bar{\eta}(\mathcal{C})$ or ∞, i.e., $\eta(\mathcal{C}) \leq rank(\mathcal{C})$ or $\eta(\mathcal{C}) = \infty$.*

The proof is given in Section 6.

Definition 3.9. The *homological connectivity* $\eta_H(\mathcal{C})$ of $\mathcal{C}$ is the minimal number k such that $\tilde{H}_k(\mathcal{C}) \neq 0$, plus 1.

Here $\tilde{H}_k(\mathcal{C})$ is the k-th reduced homological group of $\mathcal{C}$ with rational coefficients. Since the audience of this survey is meant to be combinatorialists, we shall not define it here. Let us just say something vague - homology is the algebra of holes, and $\tilde{H}_k(X)$ counts the number of holes of dimension $k+1$, namely copies of the sphere S^k that are not filled in the complex. In particular, $\tilde{H}_k = 0$ means that there are no holes whose boundaries are of dimension k.

Here is a basic fact about the connection between the two versions of connectivity.

Theorem 3.10 (e.g., [26])**.**
(i) $\eta_H(\mathcal{C}) \geq \eta(\mathcal{C})$.
(ii) (Hurewicz' theorem) If $\eta(\mathcal{C}) \geq 3$ then $\eta_H(\mathcal{C}) = \eta(\mathcal{C})$.

The condition $\eta(\mathcal{C}) \geq 3$ is indeed necessary: this can be shown by a complex called "Poincaré homology sphere" [38, Section 18], denoted below $\mathcal{P}$. (The homology sphere is a compact 3-manifold, thus can be triangulated [37].) This simplicial complex has the same homology as S^3, and therefore $\eta_H(\mathcal{P}) = \eta_H(S^3) = 4$, however, the fundamental group of $\mathcal{P}$ in non-trivial, implying $\eta(\mathcal{P}) = 2$. A simpler example is an acyclic space in [26, Example 2.38], in which the ith homological group is trivial for every i so that $\eta_H = \infty$, while the fundamental group is non-trivial so that $\eta = 2$.

Recalling the join operation in Section 2.4.2, a fact that we shall not prove here is the following.

Theorem 3.11. *For any two abstract complexes $\mathcal{C}$ and $\mathcal{D}$, $\eta_H(\mathcal{C} * \mathcal{D}) = \eta_H(\mathcal{C}) + \eta_H(\mathcal{D})$.*

This is yet another testimony to the simplicity of formulations, when using "η" rather than "connectivity". It is quite easy to see that if $\eta(\mathcal{C}) = \eta(\mathcal{D}) = 1$, i.e., $\mathcal{C}$ and $\mathcal{D}$ are both nonempty and disconnected, then $\eta_H(\mathcal{C} * \mathcal{D}) = \eta(\mathcal{C} * \mathcal{D}) = 2$. Also, if at least one of $\mathcal{C}$ and $\mathcal{D}$ is connected and the other is nonempty then $\eta(\mathcal{C} * \mathcal{D}) \geq 3$. Together with Theorem 3.10 this yields the following two results.

Theorem 3.12. *If $\mathcal{C}$ and $\mathcal{D}$ are nonempty abstract complexes then $\eta(\mathcal{C} * \mathcal{D}) = \eta_H(\mathcal{C} * \mathcal{D})$.*

Theorem 3.13. *For any two abstract complexes $\mathcal{C}$ and $\mathcal{D}$, $\eta(\mathcal{C} * \mathcal{D}) \geq \eta(\mathcal{C}) + \eta(\mathcal{D})$.*

Example 3.14. Let $*^k S^0$ be $S^0 * S^0 * \ldots * S^0$ k times. Writing a triangulation of the ith copy as $\{\emptyset, \{x_i\}, \{y_i\}\}$, then σ in the join complex if and only if $|\sigma \cap \{x_i, y_i\}| \leq 1$ for every $1 \leq i \leq k$. Then as discussed in Example 3.6, the join is the boundary of the k-dimensional crosspolytope, which is homeomorphic to S^{k-1}. In this case equality holds: $\eta(S^0) = 1$ and $\eta(*^k S^0) = k$.

4. Topological Hall

The "marriage theorem", commonly attributed to P. Hall and preceded in different attires by Frobenius and König, is the following.

Theorem 4.1 (Hall's marriage theorem)**.** *Let $\mathcal{A} = (V_i)_{i\in[m]}$ be a system of sets. If $|\bigcup_{i\in I} V_i| \geq |I|$ for every $I \subseteq [m]$, then $\mathcal{A}$ has a system of distinct representatives (SDR), i.e., an injective function $f : [m] \to \cup_{i\in[m]} V_i$ such that $f(i) \in V_i$ for each $i \in [m]$.*

In [13] a generalization was proved, solving a conjecture proposed in [1]. Given a family $\mathcal{S} = (S_1, \ldots, S_m)$ of (not necessarily disjoint or even distinct) sets and $I \subseteq [m]$, write $\mathcal{S}_I := \bigcup_{i\in I} S_i$. A *transversal* is a choice function, namely a (not necessarily injective) function $f : [m] \to \cup_{i\in[m]} S_i$ such that $f(i) \in S_i$ for every $i \in [m]$. Other familiar names are *rainbow set* (viewing the sets $[m]$ as "colors") and *system of representatives (SR)*. When there is no risk of confusion, we shall also call the image of f in V, rather than f itself, a transversal. If Image(f) belongs to a complex $\mathcal{C}$ we say that f is a *$\mathcal{C}$-transversal*.

As usual, the maximum size of a matching (a set of disjoint edges) in a hypergraph H is denoted by $\nu(H)$.

Theorem 4.2 ([13])**.** *Let $\mathcal{H} = (H_1, \ldots, H_m)$ be a family of r-uniform hypergraphs. If $\nu(\mathcal{H}_I) > (|I| - 1)r$ for every $I \subseteq [m]$, then $\mathcal{H}$ has a transversal consisting of disjoint edges.*

Hall's theorem is the case $r = 1$. The proof used Sperner's lemma (Lemma 1.2). The first author of the current paper then observed (cited in [35]) that a general theorem was essentially proved there, involving the notion of topological connectivity.

Theorem 4.3 (Topological Hall)**.** *Let $\mathcal{V} = (V_1, \ldots, V_m)$ be a system of sets, and let $\mathcal{C}$ be a complex on $\bigcup_{i\in[m]} V_i$. If $\eta(\mathcal{C}[\mathcal{V}_I]) \geq |I|$ for every $I \subseteq [m]$, then $\mathcal{V}$ has a $\mathcal{C}$-transversal.*

The (slightly stronger) homological version, in which η is replaced by η_H, was proved by Meshulam [35].

As with Hall's original theorem it is sometimes convenient to formulate the theorem in terms of bipartite graphs.

Theorem 4.4. *Let G be a bipartite graph with sides M and W, and let $\mathcal{C}$ be complex on W. If $\eta(\mathcal{C}[N_G(A)]) \geq |A|$ for every $A \subseteq M$, then there exists a (not necessarily $1-1$) function $f : M \to W$ contained in $E(G)$ whose image belongs to $\mathcal{C}$.*

Here $N_G(A)$ is the set of neighbors of A in G.

For $\mathcal{V} = (V_1, \ldots, V_m)$, let

$$def(\mathcal{V}) := \max_{I\subseteq[m]} |I| - \eta(\mathcal{C}[\mathcal{V}_I]).$$

Theorem 4.5 (Topological Hall, a deficiency version)**.** *There exists a $\mathcal{C}$-transversal of $|\mathcal{V}| - def(\mathcal{V})$ sets V_i.*

The proof is standard - adding join-wise $def(\mathcal{V})$ dummy S^0s (each having $\eta = 1$), which makes the deficiency 0, and then applying Topological Hall.

Remark 4.6. $\mathcal{C}$-transversals, which can be viewed as rainbow sets in $\mathcal{C}$, add another dimension to $\mathcal{C}$ itself. For example, rainbow matchings in graphs can be viewed as matchings in 3-uniform hypergraphs. This takes us to the world of NP-hard problems, so it is unlikely that the problem is co-NP, which discourages attempts to formulate necessary and sufficient conditions for the existence of such transversals. For this reason we are only looking for sufficient conditions.

5. Proof of Theorem 4.3

5.1. **A toy case:** $m = 2$. The assumption $\eta(\mathcal{C}[V_i]) \geq 1$ implies that each V_i is non-empty. Choose two points $v_1 \in V_1,\ v_2 \in V_2$. If $v_1 = v_2 = v$ then choosing v as a representative for both V_1 and V_2 is a $\mathcal{C}$-transversal. If not, the assumption that $\eta(\mathcal{C}[V_1 \cup V_2]) \geq 2$ means that there exists a $v_1 - v_2$ path in $\mathcal{C}[V_1 \cup V_2]$. This path must switch at some place from V_1 to V_2, namely it must contain an edge uv where $u \in V_1, v \in V_2$. Then $\{u, v\}$ serve as a $\mathcal{C}$-transversal.

The "crossing from V_1 to V_2" argument is just the 1-dimensional case of Sperner's lemma. For $m = k$ we shall invoke the $(k-1)$-dimensional case of the lemma.

5.2. **The general case.** For the size k, for every $I \subseteq [m]$ of size k, we inductively construct a triangulation $\mathcal{T}_I$ of Δ_I in such a way that $\mathcal{T}_J$ and $\mathcal{T}_K$ agree on $\Delta_{J \cap K}$ for every $J, K \subseteq [m]$ of size at most k. Furthermore, we construct a map $\lambda_I : V(\mathcal{T}_I) \to \mathcal{V}_I$ satisfying the following conditions:

- $\lambda_I \mid_{V(\mathcal{T}_J)} = \lambda_J$ for all $J \subseteq I$,
- and λ_I is a simplicial map.

The case $k = 0$ is vacuous. For $k = 1$, for each $i \in [m]$, we can take $\mathcal{T}_{\{i\}} = \{\Delta_{\{i\}}\}$ and let $\lambda_{\{i\}}(e_i) = v_i$ for an arbitrary $\{v_i\} \in \mathcal{C}[V_i]$ (the non-emptyness of $\mathcal{C}[V_i]$ is implied by $\eta(\mathcal{C}[V_i]) \geq 1$). This construction satisfies the required conditions.

Assuming that $\mathcal{T}_J$ has been defined for for all $J \subseteq [m]$ of size less than k, let $I \subseteq [m]$ be of size k. The triangulations $\mathcal{T}_J$ for $J \subsetneq I$ are compatible, meaning that they form together a triangulation $\mathcal{T}$ of the boundary $\partial \Delta_I \cong S^{k-2}$. We define $\tau : \cup_{J \subsetneq I} V(\mathcal{T}_J) \to V(\mathcal{C})$ as $\tau(p) = \lambda_J(p)$ if $p \in V(\mathcal{T}_J)$. The compatibility and induction hypothesis guarantee that τ is a well-defined simplicial map from $\mathcal{T}$ to $\mathcal{C}[\mathcal{V}_I]$. By the assumption $\eta(\mathcal{C}[\mathcal{V}_I]) \geq |I| = k$ in the theorem, τ can be extended to a simplicial map λ_I from a triangulation $\mathcal{T}_I$ of $\Delta_I \cong B^{k-1}$ to $\mathcal{C}[\mathcal{V}_I]$ such that $\mathcal{T}_I$ agrees with $\mathcal{T}$ on $\partial \Delta_I$ (namely not introducing new points on the boundary) and λ_I agrees with τ on $V(\mathcal{T})$. This construction satisfies the required conditions.

Let $\mathcal{K} = \mathcal{T}_{[m]}$ and $\lambda = \lambda_{[m]}$. By the construction $\mathcal{K}$ is a triangulation of $\Delta_{[m]}$ with a Sperner coloring λ. By Sperner's lemma there exists in $\mathcal{K}$ a multicolor simplex, whose image under λ is the desired $\mathcal{C}$-transversal.

Remark 5.1. Since $\bar{\eta} \leq \eta$, Theorem 4.3 is valid also when η is replaced by $\bar{\eta}$. As we shall see below (Theorem 10.1), in this version we can also demand injectivity of f.

6. Skeletons and retracts

Definition 6.1. For a set V and a natural number k, we write $\binom{V}{\leq k}$ for the abstract simplicial complex of all subsets of V of size at most k. We also write $\binom{V}{\leq \infty}$ for the abstract simplicial complex of all subsets of V.

If $\mathcal{C}$ is an abstract simplicial complex on the ground set V, then $tr_k(\mathcal{C}) = \mathcal{C} \cap \binom{V}{\leq k}$ is called the *k-truncation* or the $(k-1)$-dimensional *skeleton* of $\mathcal{C}$.

Observation 6.2 (e.g., Proposition 4.4.2 in [34]). *$\eta(\mathcal{C}) \geq k$ if and only if $\eta(tr_k(\mathcal{C})) \geq k$.*

Theorem 6.3. *If $k < |V|$, then $\eta(\binom{V}{\leq k}) = k$. If $k \geq |V|$, then $\eta(\binom{V}{\leq k}) = \infty$.*

Proof. To prove the first equality, clearly $\eta(\binom{U}{\leq k}) \geq k$ for any set U: for any triangulation $\mathcal{T}$ of S^{k-2} and any simplicial map $f : \mathcal{T} \to \binom{U}{\leq k}$, take a new point $x \notin V(\mathcal{T})$ and $v \in U$. Let $\mathcal{T}' = x * \mathcal{T}$, which is a triangulation of B^{k-1}, and let $g(x) = v$ and $g(y) = f(y)$ for $y \in V(\mathcal{T})$. Since for any $\sigma \in \mathcal{T}$, $|\sigma| \leq k - 1$ so that $g[\{x\} \cup \sigma] = \{v\} \cup f[\sigma] \in \binom{U}{\leq k}$. Thus g is a simplicial map .

For the other direction, assume for contradiction that $\eta(\binom{V}{\leq k}) > k$, and let $V_1 \dots, V_{k+1}$ be nonempty disjoint sets whose union is V. Then $\eta(\binom{\cup_{i \in I} V_i}{\leq k}) \geq |I|$ for any $I \subseteq [k+1]$, thus by Theorem 4.3, $(V_1, \dots, V_{k+1})$ has a $\binom{V}{\leq k}$-transversal, which is impossible.

The second equality is straightforward (by Example 2.2 and Example 3.4). □

Definition 6.4. Let X be a topological space and Y a subspace of X. We say that Y is a retract of X if there exists a continuous function $r : X \to Y$, where $r(y) = y$ for every $y \in Y$. In this case, we say that r is a retraction from X to Y. We say that an abstract simplicial complex $\mathcal{A}$ is a retract of an abstract simplicial complex $\mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and $||\mathcal{A}||$ is a retract of $||\mathcal{B}||$.

Observation 6.5. *If $\mathcal{A}$ is a retract of $\mathcal{B}$ then $\eta(\mathcal{A}) \geq \eta(\mathcal{B})$.*

Observation 6.6. *If $\mathcal{A}$ is a retract of $\mathcal{B}$ and $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{B}$ then $\mathcal{A}$ is a retract of $\mathcal{C}$.*

The following equivalent definition of η is sometimes more convenient.

Theorem 6.7. *Let $\mathcal{C}$ be an abstract simplicial complex on a set V and let $k \in \{1, \dots, |V|\}$. Then $\eta(\mathcal{C}) \geq k$ if and only if $\mathcal{C} \cap \binom{V}{\leq k}$ is a retract of $\binom{V}{\leq k}$.*

The "if" part follows from Observation 6.5 and Theorem 6.3 (and from Example 2.2 and Example 3.4 for the case $k = |V|$). For the "only if" part, we can repeatedly use the following lemma.

Lemma 6.8. *Let $\mathcal{C} \subseteq \mathcal{A} \subseteq \mathcal{B}$ be three abstract simplicial complexes, where $\mathcal{B} \setminus \mathcal{A} = \{\sigma\}$, i.e., $\mathcal{B}$ is obtained from $\mathcal{A}$ by adding only one simplex. Let $k = |\sigma|$. If $\mathcal{C}$ is a retract of $\mathcal{A}$ and $\eta(\mathcal{C}) \geq k$, then $\mathcal{C}$ is also a retract of $\mathcal{B}$.*

Proof. Let r be the retraction from $||\mathcal{A}||$ to $||\mathcal{C}||$. Since $\eta(\mathcal{C}) \geq k$, the restriction of r to $\partial||\sigma||$ can be extended to a continuous mapping from $||\sigma||$ to $||\mathcal{C}||$. This gives a retraction from $||\mathcal{B}||$ to $||\mathcal{C}||$. □

By repeatedly using the lemma we can prove the following result.

Theorem 6.9. *Let $\mathcal{C} \subseteq \mathcal{A} \subseteq \mathcal{B}$ be three abstract simplicial complexes. If $\eta(\mathcal{C}) \geq \max(|\sigma| : \sigma \in \mathcal{B} \setminus \mathcal{A})$ then every retraction from $\mathcal{A}$ to $\mathcal{C}$ can be extended to a retraction from $\mathcal{B}$ to $\mathcal{C}$.*

6.1. **Proof of Claim 3.8.**

Theorem 6.10. *Let $\mathcal{C}$ be an abstract simplicial complex on a set V and let $k \in \{1, \dots, |V|\}$. If some vertex $v \in V$ belongs to all faces of size k then $\eta(\mathcal{C}) \neq k$.*

Proof. Note first that in this case $\mathcal{C} = tr_k(\mathcal{C})$, otherwise there exists $\sigma \in \mathcal{C}$ with $|\sigma| = k+1$ and a subset of $\sigma \setminus \{v\}$, which is a face of size k and does not contain v, a contradiction. Assume for contradiction that $\eta(\mathcal{C}) = k$. Then by Theorem 6.7, $\mathcal{C}$ is a retract of $\mathcal{A} := \binom{V}{\leq k}$. Let $\mathcal{B} := v * (\mathcal{C} \cap \binom{V}{\leq k-1})$. Then $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{C}$ is a retract of $\mathcal{A}$, and therefore $\mathcal{C}$ is a retract of $\mathcal{B}$. But since $\mathcal{B}$ is a cone, by Observation 6.5 and Example 3.5 we get $\eta(\mathcal{C}) \geq \eta(\mathcal{B}) = \infty > k$. □

We can get Claim 3.8 as a corollary of Theorem 6.10. Indeed, the condition of Theorem 6.10 vacuously holds for every $k > rank(\mathcal{C})$, so we can have either $\eta(\mathcal{C}) \leq rank(\mathcal{C})$ or $\eta(\mathcal{C}) = \infty$.

7. A homotopical version of the Mayer-Vietoris theorem

A useful tool for obtaining lower bounds on the homological connectivity of complexes is the Mayer-Vietoris theorem, spelling the exactness of the following sequence. Given two complexes $\mathcal{A}, \mathcal{B}$, the following sequence is exact (see, e.g., [26, Section 2.2]):

$$\ldots \to H_{n+1}(\mathcal{A} \cup \mathcal{B}) \to H_n(\mathcal{A} \cap \mathcal{B}) \to H_n(\mathcal{A}) \oplus H_n(\mathcal{B}) \to H_n(\mathcal{A} \cup \mathcal{B}) \to \ldots$$

Here the union and the intersection can be interpreted either abstractly or with respect to a geometric realization. The exactness of a sequence $X \to Y \to Z$ of groups implies that if $X = Z = \{0\}$ then $Y = \{0\}$. This, and the definition of η_H, yield the following:

Theorem 7.1. *For any two complexes $\mathcal{A}, \mathcal{B}$ we have:*
- *(i)* $\eta_H(\mathcal{A}) \geq \min(\eta_H(\mathcal{A} \cup \mathcal{B}), \eta_H(\mathcal{A} \cap \mathcal{B}))$
- *(ii)* $\eta_H(\mathcal{A} \cup \mathcal{B}) \geq \min(\eta_H(\mathcal{A}), \eta_H(\mathcal{B}), \eta_H(\mathcal{A} \cap \mathcal{B}) + 1)$
- *(iii)* $\eta_H(\mathcal{A} \cap \mathcal{B}) \geq \min(\eta_H(\mathcal{A}), \eta_H(\mathcal{B}), \eta_H(\mathcal{A} \cup \mathcal{B}) - 1)$.

The corresponding homotopic analogues of the first two inequalities still hold. However, the homotopic analogue of the third inequality does not.

Theorem 7.2. *For any two complexes $\mathcal{A}, \mathcal{B}$ we have:*
- *(i)* $\eta(\mathcal{A}) \geq \min(\eta(\mathcal{A} \cup \mathcal{B}), \eta(\mathcal{A} \cap \mathcal{B}))$
- *(ii)* $\eta(\mathcal{A} \cup \mathcal{B}) \geq \min(\eta(\mathcal{A}), \eta(\mathcal{B}), \eta(\mathcal{A} \cap \mathcal{B}) + 1)$
- *(iii) There exist complexes $\mathcal{A}$ and $\mathcal{B}$ such that* $\eta(\mathcal{A} \cap \mathcal{B}) < \min(\eta(\mathcal{A}), \eta(\mathcal{B}), \eta(\mathcal{A} \cup \mathcal{B}) - 1)$.

Proof of part (i). Let $V = V(\mathcal{A} \cup \mathcal{B})$. We may first assume that

$$k = \min(\eta(\mathcal{A} \cup \mathcal{B}), \eta(A \cap \mathcal{B}))$$

is finite, therefore $k \leq |V|$ by Claim 3.8. Let $\mathcal{C} = \binom{V}{\leq k}$. By Observation 6.2 and Theorem 6.7, there exist a retraction f from $||\mathcal{C}||$ to $||tr_k(\mathcal{A} \cup \mathcal{B})|| = ||tr_k(\mathcal{A}) \cup tr_k(\mathcal{B})||$ and a retraction g from $\mathcal{C}$ to $||tr_k(\mathcal{A} \cap \mathcal{B})|| = ||tr_k(\mathcal{A}) \cap tr_k(\mathcal{B})||$. We can now define a retraction h from $||\mathcal{C}||$ to $||tr_k(\mathcal{A})||$ by taking $h(x) = f(x)$ if $f(x) \in ||tr_k(\mathcal{A})||$ and $h(x) = g(f(x))$ if $f(x) \in ||tr_k(\mathcal{B})||$. This is a continuous function because $f^{-1}(||tr_k(\mathcal{A})||)$ and $f^{-1}(||tr_k(\mathcal{B})||)$ are both closed sets and the definitions agree on the intersection of these two sets. Therefore by Observation 6.5 and Theorem 6.3, $\eta(tr_k(\mathcal{A})) \geq \eta(\mathcal{C}) \geq k$, which by Observation 6.2 implies $\eta(\mathcal{A}) \geq k$.

If $\min(\eta(\mathcal{A} \cup \mathcal{B}), \eta(A \cap \mathcal{B})) = \infty$, the similar argument holds for any

$$k \geq \min(\eta(\mathcal{A} \cup \mathcal{B}), \eta(A \cap \mathcal{B}), |V|),$$

which implies $\eta(\mathcal{A}) = \infty$. □

Proof of part (ii). If $\eta(\mathcal{A}\cup\mathcal{B}) \geq 3$, the inequality holds by Hurewicz's theorem ((ii) in Theorem 3.10), (ii) in Theorem 7.1, and (i) in Theorem 3.10.

Thus it remains to prove the inequality for $\eta(\mathcal{A}\cup\mathcal{B}) \leq 2$.

If $\eta(\mathcal{A}\cup\mathcal{B}) = 0$, then $\eta(\mathcal{A}) = \eta(\mathcal{B}) = 0$ and the inequality holds.

If $\eta(\mathcal{A}\cup\mathcal{B}) = 1$, then if $\eta(\mathcal{A}\cap\mathcal{B}) = 0$, the inequality holds. Thus we may assume $\eta(\mathcal{A}\cap\mathcal{B}) \geq 1$, i.e., $||\mathcal{A}\cap\mathcal{B}|| \neq \emptyset$. If $\eta(\mathcal{A}) = 1$ or $\eta(\mathcal{B}) = 1$, then the inequality holds. The remaining case is that both $||\mathcal{A}||$ and $||\mathcal{B}||$ are path-connected. But then $||\mathcal{A}\cup\mathcal{B}||$ is path-connected so that $\eta(\mathcal{A}\cup\mathcal{B}) \geq 2$: for any $x, y \in ||\mathcal{A}\cup\mathcal{B}||$, taking $z \in ||\mathcal{A}\cap\mathcal{B}||$, there is a path between x and z and a path between z and y by the path-connectedness of $|\mathcal{A}||$ and $||\mathcal{B}||$. It contradicts with the assumption $\eta(\mathcal{A}\cup\mathcal{B}) = 1$.

If $\eta(\mathcal{A}\cup\mathcal{B}) = 2$, we only need to exclude the case that $\eta(\mathcal{A}\cap\mathcal{B}) \geq 2$, $\eta(\mathcal{A}) \geq 3$, and $\eta(\mathcal{B}) \geq 3$. If it is the case, we shall prove that any continuous map $f : S^1 \to ||\mathcal{A}\cup\mathcal{B}||$ can be extended to a continuous map $g : B^2 \to ||\mathcal{A}\cup\mathcal{B}||$. It implies that $\eta(\mathcal{A}\cup\mathcal{B}) \geq 3$, a contradiction.

First note that if $f[S^1] \subseteq ||\mathcal{A}||$ or $f[S^1] \subseteq ||\mathcal{B}||$, then f can be extended by the connectivity condition of $\mathcal{A}$ and $\mathcal{B}$. Thus there exist $a, b \in S^1$ such that $f(a) \in ||\mathcal{A}|| \setminus ||\mathcal{B}||$ and $f(b) \in ||\mathcal{B}|| \setminus ||\mathcal{A}||$. Furthermore, since $||\mathcal{A}||$ and $||\mathcal{B}||$ are closed sets, there exist distinct $x, y \in S^1$ such $f(x), f(y) \in ||\mathcal{A}\cap\mathcal{B}||$. Note that x and y divide the S^1 into two segments, say I_1 and I_2. Then by path-connectedness of $||\mathcal{A}\cap\mathcal{B}||$, there exists a continuous map $h : [0, 1] \to ||\mathcal{A}\cap\mathcal{B}||$, whose image is a path in $||\mathcal{A}\cap\mathcal{B}||$ connecting $h(0) = f(x)$ and $h(1) = f(y)$. Attaching the interval $I := [0, 1]$ to the $S^1 = I_1 \cup I_2$ by identifying 0 with x and 1 with y, we obtain two S^1: one is $S_1 := I_1 \cup I$ and the other is $S_2 := I_2 \cup I$. For $i \in \{1, 2\}$, define $f_i : S_i \to ||\mathcal{A}\cup\mathcal{B}||$ by $f_i(z) = f(z)$ if $z \in I_i$ and $f_i(z) = h(z)$ if $z \in I$. Compared to f, the map f_1 (and f_2) has one less point mapped to $||\mathcal{B}|| \setminus ||\mathcal{A}||$ (and $||\mathcal{A}|| \setminus ||\mathcal{B}||$, respectively). Then by the compactness of S^1 and by induction, f_i can be extended to a continuous map $g_i : B^2 \to ||\mathcal{A}\cup\mathcal{B}||$ for $i \in \{1, 2\}$. Since g_1 and g_2 agree on the common segment I, they induce a continuous map $g : B^2 \to ||\mathcal{A}\cup\mathcal{B}||$ extending f, as claimed.

□

Proof of part (iii). Let $\mathcal{C}$ be some simplicial complex with $\eta_H(\mathcal{C}) > \eta(\mathcal{C}) = 2$ (for example the Poincaré Sphere mentioned below Theorem 3.9). Let $\mathcal{D}$ be some disconnected nonempty simplicial complex (for example we can take $||\mathcal{D}|| = S^0$), and write $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ where $\mathcal{D}_1$ and $\mathcal{D}_2$ are nonempty and $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$. Let $\mathcal{A} = \mathcal{C} * \mathcal{D}_1$ and $\mathcal{B} = \mathcal{C} * \mathcal{D}_2$. Then $\mathcal{A}\cup\mathcal{B} = \mathcal{C} * \mathcal{D}$. Using Theorems 3.13 and 3.12, we get

$$\begin{aligned} \eta(\mathcal{A}) = \eta_H(\mathcal{A}) = \eta_H(\mathcal{C}) + \eta_H(\mathcal{D}_1) \geq 3 + 1 = 4, \\ \eta(\mathcal{B}) = \eta_H(\mathcal{B}) = \eta_H(\mathcal{C}) + \eta_H(\mathcal{D}_2) \geq 3 + 1 = 4, \\ \eta(\mathcal{A}\cup\mathcal{B}) = \eta_H(\mathcal{A}\cup\mathcal{B}) = \eta_H(\mathcal{C}) + \eta_H(\mathcal{D}) \geq 3 + 1 = 4, \end{aligned}$$

but

$$\eta(\mathcal{A}\cap\mathcal{B}) = \eta(\mathcal{C}) = 2 < \min(\eta(\mathcal{A}), \eta(\mathcal{B}), \eta(A\cup\mathcal{B}) - 1),$$

as the claim. □

Corollary 7.3. *For a complex* $\mathcal{C}$ *and* $x \in V(\mathcal{C})$ *we have*

$$\eta(\mathcal{C}) \geq \min(\eta(\mathcal{C} - x),\ \eta(\mathrm{lk}_{\mathcal{C}}(x)) + 1).$$

Proof. Let $\mathcal{A} = \mathcal{C} - x$ and $\mathcal{B} = x * \mathrm{lk}_{\mathcal{C}}(x)$ ($x * \mathrm{lk}_{\mathcal{C}}(x)$ is called the *star* of x and is denoted by $star(x)$). Then $\mathcal{A} \cap \mathcal{B} = \mathrm{lk}_{\mathcal{C}}(x)$ and $\mathcal{A} \cup \mathcal{B} = \mathcal{C}$. Applying (2) of Theorem 7.2 and Theorem 3.13 yields the desired claim. $\square$

8. Flag complexes — the independence complexes of graphs

To apply Theorem 4.3 we need combinatorially defined lower bounds on the connectivity of complexes. We start with bounds on the connectivity of so-called "flag complexes". A complex $\mathcal{C}$ is called a *flag complex* if $\mathcal{C}$ is *2-determined*, namely if for any $S \subseteq V(\mathcal{C})$, if $\binom{S}{2} \subseteq \mathcal{C}$, then $S \in \mathcal{C}$. For every graph G, the *independence complex* $\mathcal{I}(G)$ (namely the collection of independent sets in G) is a flag complex on $V(G)$. Conversely, every flag complex $\mathcal{C}$ is the independence complex $\mathcal{I}(G)$ of a graph G, whose vertex set is $V(\mathcal{C})$ and edges are the pairs not belonging to $\mathcal{C}$.

Most known lower bounds on $\eta(\mathcal{I}(G))$ are formulated in terms of *domination parameters*. Some are defined below. The *open neighborhood* $N(v)$ of a vertex v in a graph G is the set of neighbors of v. The *closed neighborhood* is $\bar{N}(v) = N(v) \cup \{v\}$. For a set S of vertices, let $N(S) = \bigcup_{v \in S} N(v)$ and $\bar{N}(S) = \bigcup_{v \in S} \bar{N}(v) = N(S) \cup S$.

A set D of vertices in a graph G is said to *dominate* a set S of vertices if $\bar{N}(D) \supseteq S$, and to *strongly dominate* S if $N(D) \supseteq S$. A set D is *dominating* (resp. *strongly dominating*) if it dominates (resp. strongly dominates) $V(G)$.

Here are some domination parameters (S denotes a general set of vertices):

(i) $\gamma(S, G)$ is the minimum size of a set dominating S in G.
(ii) $\gamma(V(G), G)$, namely the minimum size of a set dominating $V(G)$, is denoted by $\gamma(G)$.
(iii) $\tilde{\gamma}(G)$ is the minimum size of a set strongly dominating $V(G)$.
(iv) $\gamma^E(G)$ is the minimum number of edges whose union dominates $V(G)$.
(v) $\gamma^i(G)$ is the maximum, over all independent sets I, of $\gamma(I, G)$.

If G contains an isolated vertex, then there are no strongly dominating sets and no dominating edge sets; in this case, $\tilde{\gamma}(G)$ and $\gamma^E(G)$ are defined as ∞.

8.1. The Meshulam bound.

For a graph G and an edge $e = \{x, y\}$ of G, let $G - e$ be obtained from G by removing the edge e, and let $G \multimap e$ be the graph $G - N(\{x, y\})$, i.e., $G[V(G) \setminus N(\{x, y\})]$.

In [36] the following was proved:

Theorem 8.1 ([36])**.** $\eta(\mathcal{I}(G)) \geq \min\Big(\eta(\mathcal{I}(G - e)),\ \eta(\mathcal{I}(G \multimap e) + 1\Big)$.

Proof. Suppose $e = \{x, y\}$. Define two complexes, $\mathcal{C} := \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$ and $\mathcal{D} := \{\emptyset, \{x\}, \{y\}\}$. Set $\mathcal{A} := \mathcal{I}(G)$ and $\mathcal{B} := \mathcal{I}(G \multimap e) * \mathcal{C}$. We have

$$\mathcal{A} \cup \mathcal{B} = \mathcal{I}(G - e) \quad \text{and} \quad \mathcal{A} \cap \mathcal{B} = \mathcal{I}(G \multimap e) * \mathcal{D}.$$

Applying Theorem 3.13 and the observation following Definition 3.3, we have

$$\eta(\mathcal{I}(G \multimap e) * \mathcal{D}) \geq \eta(\mathcal{I}(G \multimap e)) + \eta(\mathcal{D}) = \eta(\mathcal{I}(G \multimap e)) + 1.$$

Now applying (i) in Theorem 7.2 completes the proof. $\square$

Theorem 8.2. *For a graph G,*

(i) $\eta(\mathcal{I}(G)) \geq \gamma^i(G)$.
(ii) $\eta(\mathcal{I}(G)) \geq \gamma^E(G)$.

The original proof of Part (i), in [13], was quite involved. Since then, some shorter and more illuminating proofs have been offered.

Proof of Theorem 8.2. Both parts of Theorem 8.2 follow from Theorem 8.1, by induction on $|E(G)|$.

To prove part (i), we may assume that there is no isolated vertex v in G, otherwise $\eta(\mathcal{I}(G)) = \eta(v * \mathcal{I}(G[V \setminus \{v\}])) = \infty$. Choose an independent set I such that $\gamma(I,G) = \gamma^i(G)$ and take an edge e incident with I. Since $I \in \mathcal{I}(G) \subseteq \mathcal{I}(G-e)$,

$$\gamma^i(G-e) \geq \gamma(I, G-e) \geq \gamma(I,G) = \gamma^i(G).$$

On the other hand, since $I \setminus (I \cap N_G(e)) \in \mathcal{I}(G \neg e)$,

$$\gamma^i(G \neg e) \geq \gamma(I \setminus (I \cap N_G(e)), G \neg e).$$

Let S be a set of minimum size that dominates $I \setminus (I \cap N_G(e))$ in $G \neg e$. And let v be the endpoint of e that is not in I. By the minimum property of S, the vertex v is not in S. While $S \cup \{v\}$ dominates I in G, which implies

$$\gamma(I \setminus (I \cap N_G(e)), G \neg e) + 1 \geq \gamma(I,G) = \gamma^i(G).$$

Therefore we prove that

$$\gamma^i(G) \leq \min\Big(\gamma^i(G-e),\ \gamma^i(G \neg e) + 1\Big)$$

and hence the desired inequality follows by induction and Theorem 8.1.

To prove part (ii), for any edge e, note that

$$\gamma^E(G) \leq \min(\gamma^E(G-e),\ \gamma^E(G \neg e) + 1).$$

Then applying Theorem 8.1 and induction completes the proof. □

Obviously, $\gamma \leq \tilde{\gamma} \leq 2\gamma^E$: Given a set of k edges whose union dominates G, the union of these edges contains at most $2k$ vertices, and it strongly dominates G, since the vertices in each edge dominate each other. Therefore Part (ii) of Theorem 8.2 implies the following result.

Theorem 8.3. $\eta(\mathcal{I}(G)) \geq \tilde{\gamma}(G)/2$.

In [12] another proof was given, using a fact about triangulations that is of interest in itself.

Theorem 8.4 (Lemma 1.2 in [12])**.** *Any triangulation $\mathcal{T}$ of S^k has an extension to a triangulation of B^{k+1}, by starting from the 1-skeleton of $\mathcal{T}$, and adding a point in the interior of the ball at a time, connected to at most $2k+2$ previous points.*

We skip the proof here.

Question 8.5. Is there a function $f(k)$ such that for every triangulation of S^k, if we add points that are connected each to $f(k)$ previous points, the process must end with the 1-skeleton of a triangulation of B^{k+1}?

Proof of of Theorem 8.3 from Theorem 8.4. We assume $\tilde{\gamma}(G) \geq 2k-1$, that is, no set of size $2k-2$ strongly dominates G. We prove that $\eta(\mathcal{I}(G)) \geq k$.

For any $-1 \leq \ell \leq k-2$, any triangulation $\mathcal{T}$ of S^ℓ, and any simplicial map $\psi : V(\mathcal{T}) \to V(\mathcal{I}(G))$, we shall find a triangulation $\mathcal{T}'$ of $B^{\ell+1}$ extending $\mathcal{T}$ and a simplicial map $\psi' : V(\mathcal{T}') \to V(\mathcal{I}(G))$ extending ψ. Let $\mathcal{T}'$ be a triangulation of B^{k-1} satisfying Theorem 8.4 and $x_1, x_2, \ldots, x_q$ be a sequence of new points there.

We define the image of x_i under ψ' one by one. Having defined $\psi'(x_1),\dots,\psi'(x_i)$, by Theorem 8.4 x_{i+1} is connected to at most $2(k-2)+2 = 2k-2$ points in $\{x_1,\dots,x_i\}\cup V(\mathcal{T})$, namely $z_1,\dots,z_t$. Hence, since no set of $2k-2$ vertices strongly dominates $V(G)$, there exists a vertex u of G that is not connected in G to any of the vertices of $\{\psi'(z_1),\dots,\psi'(z_t)\}$. Setting $\psi'(x_{i+1}) := u$ keeps the condition that the simplices of $\mathcal{T}'$ are mapped to simplices in $\mathcal{I}(G)$. Hence the map ψ' obtained in this way is simplicial from $V(\mathcal{T}')$ to $V(\mathcal{I}(G))$ as required. □

If G is the line graph $L(H)$ for an r-uniform hypergraph H, then by part (ii) of Theorem 8.2, $\eta(\mathcal{I}(G)) \geq \gamma^E(G) \geq \frac{\tau(H)}{2r-1}$, where $\tau(H)$ is the covering number of H (the minimum size of a vertex subset such that each edge of H is incident to one of the vertex). This proves a theorem proved combinatorially in [27].

Theorem 8.6 ([27])**.** *Let $\mathcal{H} = (H_1,\dots,H_m)$ be a family of r-uniform hypergraphs. If $\tau(\mathcal{H}_I) > (2r-1)(|I|-1)$ for every $I \subseteq [m]$, then $\mathcal{H}$ has a full rainbow matching.*

Conjecture 8.7. *If H is r-partite then $\eta(\mathcal{I}(L(H))) \geq \frac{\tau(H)}{2r-2}$.*

For $r = 2$ this is true by the inequality $\eta \geq \nu/2$, where $\nu(H)$ is the matching number of H, i.e., the maximum number of disjoint edges (see [3, Theorem 6.5]) and the equality $\tau = \nu$. For $\eta = 1$ this is the Gyarfas-Lehel conjecture.

9. The Γ function

Lovász defined a function on graphs, which he denoted by θ, that uses vector representation of graphs to imitate $\alpha(G)$, the independence number of the graph G. An assignment of a vector $P(v) \in \mathbb{R}^d$ for every vertex v is a representation if $P(u)\cdot P(v) = 0$ whenever $uv \in E(G)$, and $\theta(G)$ is the maximum over all representations of a measure of how hard it is to capture all vectors $P(v)$. There is a similar function, that imitates γ, the domination number. It is denoted by $\Gamma(G)$. It is defined by a different type of representation: an assignment of a vector $P(v)$ to every vertex v, satisfying the conditions that $P(u)\cdot P(v) \geq 0$ for every pair u, v of vertices, and if $uv \in E(G)$ then $P(u)\cdot P(v) \geq 1$. Domination is now done by non-negative linear combinations of vectors: a vector $\vec{u} = \sum_{v\in V(G)} \alpha(v)P(v)$ is said to dominate P if $\vec{u}\cdot P(v) \geq 1$ for every $v \in V(G)$. The value $|P|$ of P is the minimum of $\sum_{v\in V(G)} \alpha(v)$ over all dominating vectors $\vec{u}$ as above. Finally, $\Gamma(G)$ is the supremum of $|P|$ over all representations P. In [7] the following lower bound on η was proved.

Theorem 9.1. *$\eta(\mathcal{I}(G)) \geq \Gamma(G)$.*

The *canonical* vector representation P of a graph G is defined by $P(v) = \chi_v$, the incidence vector of v in $\mathbb{R}^{E(G)}$, for every vertex v.

For a hypergraph H, the *fractional width* $w^*(H)$ is the minimum of $\sum_{S\in H} f(S)$ over all $f : H \to \mathbb{R}_{\geq 0}$ with the property that for every edge $S \in H$,

$$\sum_{T\in H} f(T)\cdot |T \cap S| \geq 1. \tag{3}$$

For a function $g : H \to \mathbb{R}_{\geq 0}$ satisfying (3) for every edge S of a hypergraph H and for the canonical vector representation P of the line graph $G = L(H)$, setting $\sum_{e\in H} g(e)P(e)$ yields $|P| = w^*$. Therefore

$$\Gamma\Big(L(H)\Big) \geq w^*(H). \tag{4}$$

On the other hand, the LP dual of fractional width of a hypergraph H is the maximum of $\sum_{S\in H} g(S)$ over all $g : H \to \mathbb{R}_{\geq 0}$ with the property that for every edge $S \in H$,

$$\sum_{T\in H} g(T)\cdot |T\cap S| \leq 1.$$

A function $f : H \to \mathbb{R}_{\geq 0}$ is a *fractional matching* of a hypergraph H if for every $v \in V(H)$, $\sum_{e\in H: v\in e} f(e) \leq 1$. The minimum $f[H]$ over all fractional matching f of H is denoted by $\nu^*(H)$. Take a fractional matching f of H attaining this minimum. Suppose H is r-uniform. Set $g = \frac{f}{r}$. Then $\sum_{S\in H} g(S) = \frac{\nu^*(H)}{r}$ and for every $S \in H$,

$$\sum_{T\in H} g(T)\cdot |T\cap S| = \sum_{v\in S}\Big(\sum_{T\in H: v\in T} g(T)\Big) \leq r\cdot\frac{1}{r} = 1.$$

Therefore for any r-uniform hypergraph H,

$$w^*(H) \geq \frac{\nu^*(H)}{r}. \tag{5}$$

Combining Theorem 9.1, (4), and (5), it follows that for an r-uniform hypergraph H,

$$\eta\Big(\mathcal{M}(H)\Big) = \eta\Big(\mathcal{I}\big(L(H)\big)\Big) \geq \Gamma\Big(L(H)\Big) \geq w^*(H) \geq \frac{\nu^*(H)}{r},$$

which by Theorem 4.3 implies a strengthening of Theorem 4.2.

Theorem 9.2 ([7])**.** *Let $\mathcal{H} = (H_1, \ldots, H_m)$ be a family of r-uniform hypergraphs. If $\nu^*(\mathcal{H}_I) > (|I|-1)r$ for every $I \subseteq [m]$, then $\mathcal{H}$ has a full rainbow matching.*

10. Duplicating vertices and injectivity

In many combinatorial applications of Theorem 4.3 the transversal is required to be injective. In Hall's original theorem, for example, injectivity is the entire point. It turns out that to obtain injectivity all you need is to add to the condition $\eta(\mathcal{S}_I) \geq |I|$ the obviously necessary Hall's condition, $rank(\mathcal{S}_I) \geq |I|$. Recalling the definition of $\bar{\eta}$ in (2), the following is another formulation.

Theorem 10.1 (Topological Hall, an injective version)**.** *Let $\mathcal{S} = (V_1, \ldots, V_m)$ be a system of not necessarily disjoint subsets of V, and let $\mathcal{C}$ be a complex on V. If $\bar{\eta}(\mathcal{C}[\mathcal{S}_I]) \geq |I|$ for every $I \subseteq [m]$, then $\mathcal{S}$ has an injective $\mathcal{C}$-transversal.*

Classical Hall follows suit, taking $\mathcal{C}$ to be the universal complex, in which every set is a face.

As to the proof, the problem arises, of course, only if the sets V_i are not disjoint. The main tool used to attain injectivity is blow-ups, namely duplicating points, and splitting the copies of each point v among the sets containing v — this renders the sets disjoint. Two points u and v in an abstract simplicial complex $\mathcal{C}$ are called *twins* if $\mathrm{lk}_{\mathcal{C}}(u) = \mathrm{lk}_{\mathcal{C}}(v)$. It is not hard to verify that if u and v are twins, then $\mathcal{C}-u$ and $\mathcal{C}-v$ are isomorphic (the isomorphism maps u in $\mathcal{C}-v$ to v in $\mathcal{C}-u$, which maps a face to a face). Note that if u and v are distinct twins then $\{v\} \notin \mathrm{lk}_{\mathcal{C}}(v) = \mathrm{lk}_{\mathcal{C}}(u)$ and thus $\{u, v\} \notin \mathcal{C}$.

The relation of being twins is clearly an equivalence relation. We denote by $[v]_{twin}$ the equivalence class of v in this relation.

Theorem 10.2. *If $|[v]_{twin}| \geq 2$ then $\eta(\mathcal{C}) \leq \eta(\mathcal{C}-v)$.*

Proof. Let u and v be distinct twins. Let $\mathcal{A} = \mathcal{C} - v$ and let $\mathcal{B} = \mathrm{lk}_{\mathcal{C}}(v) * \{\{u\}, \{v\}, \emptyset\}$ then $\mathcal{A} \cup \mathcal{B} = \mathcal{C}$ and $\mathcal{A} \cap \mathcal{B} = \mathrm{lk}_{\mathcal{C}}(v) * \{\{u\}, \emptyset\}$. Therefore $\eta(\mathcal{A} \cap \mathcal{B}) = \infty$ and by part (i) of Theorem 7.2,

$$\eta(\mathcal{A}) \geq \min(\eta(\mathcal{A} \cup \mathcal{B}), \eta(\mathcal{A} \cap \mathcal{B})) = \min(\eta(\mathcal{C}), \infty) = \eta(\mathcal{C}),$$

which completes the proof. □

The inequality is sometimes strict, for example, for $\mathcal{C}$ a 4-cycle closed down, every vertex has a twin, and we have $\eta(\mathcal{C}) = 2$ and $\eta(\mathcal{C} - v) = \infty$ for any vertex v. Equality holds when there are more than two twins, as the following.

Theorem 10.3. *If* $|[v]_{twin}| \geq 3$ *then* $\eta(\mathcal{C}) = \eta(\mathcal{C} - v)$.

Proof. Let u, v, and w be distinct twins. Let $\mathcal{A} = \mathcal{C} - v$ and $\mathcal{B} = \mathcal{C} - u$. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, implying $\eta(\mathcal{A}) = \eta(\mathcal{B})$. We also have $\mathcal{A} \cup \mathcal{B} = \mathcal{C}$ (there is no $\sigma \in \mathcal{C}$ such that $\{u, v\} \subseteq \sigma$ as mentioned above) and $\mathcal{A} \cap \mathcal{B} = \mathcal{A} - u$. Since u has the twin w in $\mathcal{A}$, by Theorem 10.2 it follows that $\eta(\mathcal{A} \cap \mathcal{B}) \geq \eta(\mathcal{A})$ and by (ii) in Theorem 7.2 we get

$$\eta(\mathcal{C}) \geq \min(\eta(\mathcal{A}), \eta(\mathcal{B}), \eta(\mathcal{A} \cap \mathcal{B}) + 1) = \eta(\mathcal{A}),$$

which together with $\eta(\mathcal{C}) \leq \eta(\mathcal{A})$ in Theorem 10.2 completes the proof. □

For an abstract simplicial complex $\mathcal{C}$ and a function $w : V(\mathcal{C}) \to \mathbb{N}$, we construct an abstract simplicial complex $DUP_w(\mathcal{C})$ as follows. For every $v \in V(\mathcal{C})$ let $J(v)$ be a set of size $w(v)$. The points of $DUP_w(\mathcal{C})$ are all the pairs (v, j), where $v \in V(\mathcal{C})$ and $j \in J(v)$. Let the simplices of $DUP_w(\mathcal{C})$ be all sets $\{(v_1, j_1), \ldots, (v_k, j_k)\}$, where $v_1, \ldots, v_k$ are distinct, $j_\ell \in J(v_\ell)$ for each $\ell \in [k]$, and $\{v_1, \ldots, v_k\} \in \mathcal{C}$. If w is the constant function $w \equiv k$ then we write $DUP_k(\mathcal{C})$ for $DUP_w(\mathcal{C})$.

Remark 10.4. $DUP_2(\mathcal{C})$ is called the *deleted join* of $\mathcal{C}$. See, e.g., [34, Section 5.5].

Theorems 10.2 and 10.3 imply the following result.

Theorem 10.5. *For every* $U \subseteq V(\mathcal{C})$,

$$\eta(DUP_w(\mathcal{C})[\{(u, j) \mid u \in U, j \in J(u)\}]) \geq \bar{\eta}(\mathcal{C}[U]).$$

Proof of Theorem 10.1. For every $v \in V$ let $J(v) = \{i \mid v \in V_i\}$. Let $w(v) = |J(v)|$. Let $W_i = \{(v, i) \mid v \in V_i\}$. Apply now Theorem 4.3 and Theorem 10.5. □

Corollary 10.6. *Let* $\mathcal{S} = (V_1, \ldots V_m)$ *be a system of subsets of* $V(G)$*, where* G *is graph. If for every* $I \subseteq [m]$ *either* $\gamma^i(G[\mathcal{S}_I]) \geq |I|$ *or* $\gamma^E(G[\mathcal{S}_I]) \geq |I|$ *then there exists an injective* $\mathcal{I}(G)$*-transversal.*

This follows from Theorem 8.2 and the fact that a maximal (with respect to containment) independent set in a graph is dominating, and hence $\gamma^i(G) \leq \gamma(G) \leq rank(\mathcal{I}(G))$ and $\gamma^E(G) \leq \gamma(G) \leq rank(\mathcal{I}(G))$.

11. Matroids

A *matroid* is a closed down hypergraph, in which, in every induced sub-hypergraph all maximal edges are of maximum size. Matroids are known to be well-behaved with respect to matchings, and not surprisingly they are so also with respect to topological connectivity.

Theorem 11.1 (Whitney [43])**.** *If $\mathcal{M}$ is a matroid then $\eta(\mathcal{M}) \geq rank(\mathcal{M})$, with equality holding unless there is a vertex belonging to all bases, in which case $\eta(\mathcal{M}) = \infty$.*

This implies that in matroids $\bar{\eta} = rank$. Together with Theorem 4.3 this yields a famous generalization of Hall's theorem, due to Rado [39], which is a special case of Theorem 4.3.

Theorem 11.2 (Rado)**.** *Let $\mathcal{V} = (V_1, \ldots, V_m)$ be a system of sets in a matroid $\mathcal{M}$. If $rank_{\mathcal{M}}(\mathcal{V}_I) \geq |I|$ for every $I \subseteq [m]$ then there is an $\mathcal{M}$-transversal.*

A *partition matroid* is defined by a partition of the ground set, where a set is a face ("independent") if it contains at most one element from each part. Let $\mathcal{P}(Q)$ be the partition matroid with one non-empty part Q. Then $\eta(\mathcal{P}(Q)) = \infty$ if $|Q| = 1$, and $\eta(\mathcal{P}(Q)) = 1$ (namely it is disconnected) if $|Q| > 1$.

Lemma 11.3. *If $\mathcal{M}$ is a partition matroid with n non-empty parts, then $\eta(\mathcal{M}) = \infty$ if there is a part of size 1, otherwise $\eta(\mathcal{M}) = rank(\mathcal{M}) = n$.*

Proof. Naming the parts of a partition matroid $\mathcal{M}$ as P_i for $1 \leq i \leq n$, the matroid $\mathcal{M}$ is clearly the join $\mathcal{P}(P_1) * \cdots * \mathcal{P}(P_n)$. This together with Theorem 3.13 implies the lower bound. Yet another proof: Let G be the disjoint union of cliques on P_i for $1 \leq i \leq n$. Clearly, $\mathcal{M} = \mathcal{I}(G)$, and $\gamma^i(G) = n$, so Theorem 8.2 implies $\eta(\mathcal{M}) \geq n$.

To prove $\eta \leq n$ in the case that all the n parts are of size greater than 1, take a pair $u_i \neq v_i$ in each of the n parts. The complex induced on $\bigcup_{1 \leq i \leq n}\{u_i, v_i\}$ is then X^d as introduced in Example 3.6, that is a $(n-1)$-sphere, and it cannot be extended to a n-ball, since no face of it is contained in a face of dimension n. □

11.1. **Another derivation of Hall's theorem.** We can now derive Hall's Theorem 4.1 from Theorem 4.3 in yet another way (the first was given in Section 10). Replace every set V_i by the set $V'_i = \{(v, i) \mid v \in V_i\}$, thus rendering the sets V_i disjoint — each V'_i contains only pairs whose second coordinate is i. On the resulting set of pairs, define the partition matroid $\mathcal{M}$ with parts $P_v := \{(v, i) \mid v \in V_i\}$, for every fixed $v \in V$. The union of every k sets V'_i contains all parts P_v for $v \in \bigcup V_i$, so by Hall's condition it contains at least k non-empty parts. By Lemma 11.3, its connectivity in $\mathcal{M}$ is at least k. By Theorem 4.3 Hall, there is a transversal with image in $\mathcal{M}$; denote it by $\{(v_1, 1), \ldots, (v_n, n)\}$. Since $\mathcal{M}$ is a partition matroid, all the vertices v_i are distinct, so $v_1, \ldots, v_n$ is an SDR.

12. The intersection of a matroid and a simplicial complex

Definition 12.1. Given a matroid $\mathcal{M}$ and $S \subseteq V(\mathcal{M})$, the span of $S \subseteq V(\mathcal{M})$ is

$$span_M(S) = \{v \in V(\mathcal{M}) \mid \{v\} \cup \sigma \notin \mathcal{M} \text{ for some } \sigma \in \mathcal{M}[S]\}.$$

A set $S \subseteq V(\mathcal{M})$ is called spanning if $span_{\mathcal{M}}(S) = V(\mathcal{M})$.

Definition 12.2. Given a matroid $\mathcal{M}$ and a complex $\mathcal{C}$ on the same ground set, an $[\mathcal{M}, \mathcal{C}]$-*matching* is a set spanning in $\mathcal{M}$ and belonging to $\mathcal{C}$.

Such a set is large in $\mathcal{M}$ and small in $\mathcal{C}$ (since $\mathcal{C}$ is closed down, belonging to it is a measure of smallness). In the setting of Topological Hall, if the sets V_i are disjoint, they define a partition matroid $\mathcal{M}$, and a $\mathcal{C}$-transversal is then an $[\mathcal{M}, \mathcal{C}]$-matching. Topological Hall can be extended to the general setting.

For a set S of points in a matroid $\mathcal{M}$ let $\mathcal{M}.S$ be the contraction matroid of $\mathcal{M}$ to S, namely $\{\sigma \subseteq S \mid \sigma \cup \tau \in \mathcal{M}$ for every $\tau \in \mathcal{M}[S^c]\}$. The dual $\mathcal{M}^*$ of a matroid $\mathcal{M}$ consists of the complements of spanning sets in $\mathcal{M}$.

Topological Hall is a special case of:

Theorem 12.3 (Aharoni-Berger [3])**.** *If*

$$\eta(\mathcal{C}[S]) \geq rank(\mathcal{M}.S)$$

for every $S \subseteq V$ then there exists an $[\mathcal{M}, \mathcal{C}]$-matching.

The proof, modelled after a proof by Welsh of the non-topological version (in which η is replaced by rank), uses the notion of the *dual matroid* $\mathcal{M}^*$. By definition, $A \in \mathcal{M}^*$ if $V \setminus A$ is spanning in $\mathcal{M}$.

Outline of a proof of Theorem 12.3. Form a bipartite graph G, whose one side, M, is V, and the other, W, consists of two copies of V, namely $V' = \{v' \mid v \in V\}$ and $V'' = \{v'' \mid v \in V\}$. Connect every vertex $v \in M$ to both v' and v''. Endow V' with the complex $\mathcal{C}$, V'' with $\mathcal{M}^*$, and let $\mathcal{D}$ be the join $\mathcal{C} * \mathcal{M}^*$. Using Theorems 3.13 and 11.1, the condition of the theorem imply $\eta(\mathcal{D}[N_G(Z)]) \geq |Z|$ for every $Z \subseteq M$, and thus by Topological Hall there exists a $\mathcal{D}$-matching F (that is a $\mathcal{D}$-transversal with respect to $\{v\}_{v \in V = M}$). Let L be the set of points in M that are matched to V'. Then $L \in \mathcal{C}$ and $V \setminus L \in \mathcal{M}^*$, the latter meaning that L is spanning in M, which is the desired result. □

Remark 12.4. If $\mathcal{C}$ is a matroid, then the condition is also necessary.

Question: Is it true that if $\mathcal{C}$ is not a matroid, then not for all matroids is the condition necessary?

For two complexes $\mathcal{C}, \mathcal{D}$ on the same ground set, let $\nu(\mathcal{C}, \mathcal{D}) = rank(\mathcal{C} \cap \mathcal{D})$ and $\tau(\mathcal{C}, \mathcal{D}) = \min_{S \subseteq V} \eta(\mathcal{C}[S]) + \eta(\mathcal{D}[V \setminus S])$. Theorem 12.3 implies a generalization of Edmonds' matroid intersection theorem, in which $\mathcal{C}$ is also a matroid:

Theorem 12.5. *If $\mathcal{M}$ and $\mathcal{C}$ are a matroid and a complex, respectively, on the same ground set, then $\tau(\mathcal{M}, \mathcal{C}) \leq \nu(\mathcal{M}, \mathcal{C})$.*

Conjecture 12.6. *Let $\mathcal{M}, \mathcal{N}$ be two matroids and let $\mathcal{D} = \mathcal{M} \cap \mathcal{N}$. Then for any complex $\mathcal{C}$ we have $\tau(\mathcal{D}, \mathcal{C}) \leq 2\nu(\mathcal{D}, \mathcal{C})$.*

12.1. **Connectivity of the intersection of matroids.**

Theorem 12.7 ([3])**.** *Let $\mathcal{M}_i, 1 \leq i \leq k$ be matroids on the same ground set, and let $\mathcal{K} = \bigcap_{1 \leq i \leq k} \mathcal{M}_i$. Then $\eta(\mathcal{K}) \geq \frac{rank(\mathcal{K})}{k}$.*

Theorem 11.1 is a special case.

13. Leray-ness and the Kalai-Meshulam theorem

For a complex $\mathcal{C}$ on V, let $\lambda(\mathcal{C})$ be the largest k such that there exists $S \subseteq V$ satisfying that $\tilde{H}_k(\mathcal{C}[S]) \neq 0$. A complex $\mathcal{C}$ is said to be d*-Leray* if $\lambda(\mathcal{C}) \leq d-1$, i.e., $\tilde{H}_i(\mathcal{C}[S]) = 0$ for all $S \subseteq V$ and $i \geq d$. It is known that $\mathcal{C}$ is d-Leray if and only if $\tilde{H}_i(lk_{\mathcal{C}}(\sigma)) = 0$ for all $\sigma \in \mathcal{C}$ and $i \geq d$ (see, e.g., [31, Proposition 3.1]). Intuitively, recalling the connectives discussed in Section 3, $\lambda(\mathcal{K})$ be the largest dimension of a hole in any induced complex $\mathcal{K}[S]$ (0 if there is no hole).

Intuitively, topological Hall assumes having no small dimensional holes, and concludes the existence of a set that is large (spanning) in a given matroid $\mathcal{M}$ (that is the transversal matroid on $\mathcal{V}_{[m]}$ with respect to $\mathcal{V} = (V_1, \dots, V_m)$) and small in a complex $\mathcal{C}$ (namely belongs to $\mathcal{C}$).

Not having large dimensional holes has a combinatorial conclusion of a mirror type — the existence of a set that is small (belonging) in $\mathcal{M}$ and large (not belonging) in $\mathcal{C}$.

These two phenomena are connected by a combinatorial operation on complexes, called "combinatorial Alexander duality". The *Alexander dual* $D(\mathcal{C})$ of a complex $\mathcal{C}$ on ground set V is defined as $\{V \setminus \tau \mid \tau \notin \mathcal{C}\}$.

By combinatorial Alexander duality theorem (see, e.g., [19, Theorem 1.1]) and universal coefficient theorem for cohomology (see, e.g., [26, Theorem 3.2]), the following is in [30, Theorem 4.1].

Theorem 13.1. *For a complex $\mathcal{C}$ on V, if $V \notin \mathcal{C}$, then for all $-1 \leq i \leq |V| - 2$,*

$$\tilde{H}_i(D(\mathcal{C})) \cong \tilde{H}_{|V|-i-3}(\mathcal{C}).$$

Intuitively, the combinatorial Alexander duality operation translates small holes to large, and vice versa. The holes of $D(\mathcal{C})$ are the mirror images of the the holes in $\mathcal{C}$: the complex $D(\mathcal{C})$ has a hole of dimension $k+1$ if and only if $\mathcal{C}$ has a hole of dimension $|V| - k - 2$.

Theorem 13.2 (Theorem 1.6 in [30])**.** *For a d-Leray complex $\mathcal{C}$ and a matroid $\mathcal{M}$ on the common ground set V, if for every $\sigma \in \mathcal{C}$,*

$$rank_{\mathcal{M}}(V \setminus \sigma) \geq d + 1,$$

then $\mathcal{M} \setminus \mathcal{C} \neq \emptyset$.

As we mainly focus on homotopy in this paper, we shall use the following version.

Theorem 13.3. *(Kalai-Meshulam) For a complex $\mathcal{C}$ and a matroid $\mathcal{M}$ on the common ground set, if $\eta(D(\mathcal{C})[S]) \geq |S| - rank(\mathcal{M}[S])$ for every $S \subseteq V(\mathcal{C})$ with $S \notin D(\mathcal{C})$, then $\mathcal{M} \setminus \mathcal{C} \neq \emptyset$.*

Proof. For $V = \{v_1, \dots, v_m\}$, let $V' = \{v'_1, \dots, v'_m\}$ and $V'' = \{v''_1, \dots, v''_m\}$ be two disjoint copies of V, and let $\mathcal{M}'$ be a copy of $\mathcal{M}$ on V' and $\mathcal{D}$ be a copy of $D(\mathcal{C})$ on V''. Let $\mathcal{A} = \mathcal{M}' * \mathcal{D}$, $W = V' \cup V''$, and $W_i = \{v'_i, v''_i\}$ for $i \in [m]$. We shall apply Theorem 4.3 to show that there exists an $\mathcal{A}$-transversal $Z \subseteq W$ with respect to $(W_i)_{i \in [m]}$. If so, let U be the subset of V such that the copies of elements in U are in $Z \cap V'$. Then $V \setminus U \in D(\mathcal{C})$, which implies $U \notin \mathcal{C}$. Thus we prove $U \in \mathcal{M} \setminus \mathcal{C}$.

For every $I \subseteq [m]$, let $S = \{v_i \mid i \in I\}$. We have

$$\eta(\mathcal{A}[\cup_{i \in I} W_i]) = \eta(\mathcal{M}'[\{v'_i \mid i \in I\}] * \mathcal{D}[\{v''_i \mid i \in I\}]) \geq \eta(\mathcal{M}[S]) + \eta(D(\mathcal{C})[S]).$$

To apply Topological Hall, we shall prove that

$$\eta(\mathcal{M}[S]) + \eta(D(\mathcal{C})[S]) \geq |S|. \tag{6}$$

If $S \in D(\mathcal{C})$, then $D(\mathcal{C})[S]$ is contractible, so $\eta(D(\mathcal{C})[S]) = \infty$. We are done. Thus we may assume $S \notin D(\mathcal{C})$, meaning that $V \setminus S \in \mathcal{C}$. But then (6) is guaranteed by the assumption. □

The proof of Theorem 13.2 is similar by considering the homological connectivity, while to verify (6) one needs to apply the d-Leray property.

13.1. **Applications of the Kalai-Meshulam theorem.**

13.1.1. *Representablity and Collapsibility.* Let $\mathcal{F}$ be a family of sets. The *nerve* $N(\mathcal{F})$ of $\mathcal{F}$ is the simplicial complex whose groud set is $\mathcal{F}$ and whose simplices are all $\mathcal{F}' \subseteq \mathcal{F}$ such that $\cap_{S \in \mathcal{F}'} S \neq \emptyset$.

Definition 13.4. A simplicial complex $\mathcal{C}$ is d-representable if it is isomorphic to a nerve of a finite family of convex sets in $\mathbb{R}^d$.

A face σ of a complex $\mathcal{C}$ is said to be *free* if it is contained in a unique maximal face τ. Removing all faces containing σ, including σ itself, is then called an *elementary collapse*, and if $|\sigma| \leq d$ it is called an elementary d*-collapse.*

Definition 13.5. A complex $\mathcal{C}$ is called d*-collapsible* if there exists a sequence $\mathcal{C}_0 \to \mathcal{C}_1 \to \ldots \to \mathcal{C}_m = \{\emptyset\}$ of elementary d-collapses.

Theorem 13.6 ([42])**.** *Every d-representable complex is d-collapsible, and every d-collapsible complex is d-Leray.*

13.1.2. *Colorful Helly and colorful Carathéodory.*

Theorem 13.7 (Colorful Helly)**.** *Let* $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ *be* $d+1$ *finite families of convex sets in* $\mathbb{R}^d$*. If* $\cap_{1 \leq i \leq d+1} F_i \neq \emptyset$ *for all choices of* $F_1 \in \mathcal{F}_1, \ldots, F_{d+1} \in \mathcal{F}_{d+1}$*, then* $\cap_{F \in \mathcal{F}_i} F \neq \emptyset$ *for some* $i \in [d+1]$*.*

Theorem 13.8 (Colorful Carathéodory)**.** *Let* $A_1, \ldots, A_{d+1}$ *be finite sets of points in* $\mathbb{R}^d$*. If* $x \in \cap_{1 \leq i \leq d+1} conv(A_i)$*, then there exist* $a_1 \in A_1, \ldots, a_{d+1} \in A_{d+1}$ *such that* $x \in conv\{a_1, \ldots, a_{d+1}\}$*.*

These results reduce to the Helly theorem and Carathéodory theorem when all $\mathcal{F}_i$'s, and respectively, all A_i's coincide. Thereoms 13.7 and 13.8 are related by linear programming duality [17].

Proof of Theorem 13.7. For $V = \mathcal{F} = \cup_{1 \leq i \leq d+1} \mathcal{F}_i$, let $\mathcal{C} = N(\mathcal{F})$, which is d-representable. Let $\mathcal{M}$ be the partition matroid with parts $(\mathcal{F}_i)_{i \in [d+1]}$. Since $\mathcal{M} \subseteq \mathcal{C}$. Then by Theorem 13.2, there exists $\sigma \in \mathcal{C}$ such that $rank_{\mathcal{M}}(V \setminus \sigma) \leq d$, which means there exists $i \in [d+1]$ such that $\mathcal{F}_i \subseteq \sigma$, i.e., $\cap_{F \in \mathcal{F}_i} F \neq \emptyset$. □

13.1.3. *Drisko's theorem and relatives.* Include the derivation of Disko in $K_{n,n}$ from Colorful Carathéodory.

Let $N^*(k)$ be the complex consisting of the subsets A of $E(K_\infty)$ satisfying $\nu(A) < k$, and $BN^*(k)$ the complex consisting of the subsets A of $E(K_{\infty,\infty})$ satisfying $\nu(A) < k$.

Theorem 13.9. $N^*(k)$ *is* $(2k-1)$*-Leray, and* $BN^*(k)$ *is* $(2k-2)$*-Leray.*

Together with the Kalai-Meshulam theorem this yields a result first proved in the special case of Latin rectangles by Drisko [22], and in the general case in [4]:

Corollary 13.10. $2n-1$ *matchings of size* n *in a bipartite graph have a rainbow matching of size* n*.*

Several proofs have been given to this result. In particular, a proof using topological Hall directly yields a stronger version:

Theorem 13.11 ([6])**.** *Any sequence* $(F_i)_{1 \leq i \leq 2n-1}$ *of bipartite graph matchings, where* $|F_i| \geq \min(i, n)$ *for all* $i \leq 2n-1$*, has a rainbow matching of size* n*.*

The same question for general graphs is still open. In [5] it was proved that $3n-2$ matchings of size n in any graph have a rainbow matching of size n, and in [11] it was proved that $3n-3$ matchings of size n suffice.

Conjecture 13.12. *$2n$ matchings of size n in any graph have a rainbow matching of size n.*

This is best possible, as shown by what is called in [11] a *badge* B_n. Its vertex set is $[2n]$, the matching consisting of the even edges on the cycle C_{2n} repeats $n-1$ times, and so does the matching consisting of odd edges. To this is added a $2n-1$th matching, consisting of the edges $13, 24, 57, 68, \ldots, (2n-3)(2n-1), (2n-2)2n$.

Possibly the analogue of Theorem Theorem 13.11 is also true:

Conjecture 13.13. *Any sequence $(F_i)_{1\le i\le 2n}$ of general graphs matchings, where $|F_i| \ge \min(i,n)$ for all $1 \le i \le 2n$, has a rainbow matching of size n.*

In [15] it was proved that the complex N^k of those sets $F \subseteq K_\infty$ satisfying $\nu^*(F) < k$ is $2(k-1)$-collapsible, and hence $2(k-1)$-Leray. Together with the Kalai-Meshulam theorem this yields a fractional version of Conjecture 13.12:

Theorem 13.14 ([15])**.** *$2n$ graphs with $\nu^* = n$ have a rainbow graph with $\nu^* = n$.*

Here ν^* is the fractional matching number.

14. Colorings (=Decomposition)

For a complex $\mathcal{C}$ a $\mathcal{C}$-*cover* is a collection of faces covering $V(\mathcal{C})$. Replacing faces by their subsets, a cover produces a decomposition (in which the covering faces are disjoint). By $\beta(\mathcal{C})$ we denote the minimal size (number of faces) of a $\mathcal{C}$-cover. Also let

$$\Xi(\mathcal{C}) := \max_{S\subseteq V} \lceil \frac{|S|}{\bar{\eta}(\mathcal{C}[S])} \rceil.$$

Let $\Delta(G)$ be the maximum degree of a graph G. The bound $\eta(\mathcal{I}(G)) \ge \tilde{\gamma}/2$ in Theorem 8.3, together with the fact that $\Delta(G)\tilde{\gamma}(G) \ge |V(G)|$ and Turán's theorem $rank(\mathcal{I}(G)) \ge \frac{|V(G)|}{\Delta(G)+1}$, implies the following result.

Observation 14.1. *For any graph G with $\Delta(G) \ge 1$, $\Xi(\mathcal{I}(G)) \le 2\Delta(G)$.*

It is proved in [14] that this bound is attained if and only if G contains a copy of the complete bipartite graph $K_{\Delta(G),\Delta(G)}$.

Theorem 14.2. $\beta(\mathcal{C}) \le \lceil \Xi(\mathcal{C}) \rceil$.

Proof. Let $\lceil \Xi(\mathcal{C}) \rceil = k$. For every $v \in V = V(\mathcal{C})$, let $(v,1), \ldots, (v,k)$ be k copies of v, and let $V_v = \{(v,1), \ldots, (v,k)\}$. For every $1 \le i \le k$ let $\mathcal{C}_i$ be a copy of $\mathcal{C}$, consisting of $\sigma^i = \{(u_1,i), \ldots, (u_t,i)\}$ for each $\sigma = \{u_1, \ldots, u_t\} \in \mathcal{C}$, and let $\{(v,i) | v \in V\} = V(\mathcal{C}_i)$. Let $\mathcal{K}$ be the join $*_{i\in[k]}\mathcal{C}_i$. Then for every subset A of V, the complex $\mathcal{L}_A := \mathcal{K}[\bigcup_{a\in A} V_a]$ satisfies $\eta(\mathcal{L}_A) \ge k\eta(\mathcal{C}[A]) \ge |A|$ (the first inequality following from Theorem 3.13 and the second inequality following from the definition of k). By Theorem 4.3 there is a $\mathcal{K}$-transversal τ with respect to $(V_a)_{a\in V}$. Let $\tau_i = \tau \cap V(\mathcal{C}_i)$ for each $i \in [k]$. Viewing τ_i as a subset of $V(\mathcal{C})$ we have $\tau_i \in \mathcal{C}$, and $\bigcup_{1\le i\le k} \tau_i = V$. This proves the theorem. □

In matroids equality holds, since the inverse inequality follows from the definition of Δ and the fact that $\bar{\eta} = rank$. The theorem, combined with Observation 14.1, yields $\chi(G) \le 2\Delta(G)$ - "half" of the well-known bound, $\chi(G) \le \Delta(G) + 1$.

Problem 14.3. Is it possible to prove the last fact using topology?

Here are some conjectures concerning β:

Conjecture 14.4. *If $\mathcal{M}_1, \mathcal{M}_2$ are matroids, then* $\beta(\mathcal{M}_1 \cap \mathcal{M}_2) \leq \max(\beta(\mathcal{M}_1), \beta(\mathcal{M}_2) + 1)$

Examples are given in [10] in which $\beta(\mathcal{M}_1) = \beta(\mathcal{M}_2) = k$ and $\beta(\mathcal{M}_1 \cap \mathcal{M}_2) = k + 1$, for any positive integer $k > 1$.

Theorems 12.7, 12.3, and 14.2 imply $\beta(\mathcal{M}_1 \cap \mathcal{M}_2) \leq 2 \max(\beta(\mathcal{M}_1), \beta(\mathcal{M}_2))$.

Conjecture 14.5. *For every graph G and matroid $\mathcal{M}$ on the same ground set, $\beta(\mathcal{I}(G) \cap \mathcal{M}) \leq \max(2\Delta(G), \Xi(\mathcal{M}))$.*

This conjecture is called in [9] the "strong matroidal coloring conjecture". The case in which $\mathcal{M}$ is a partition matroid is known as the "strong coloring conjecture". In all cases in which equality occurs in Conjecture 14.5 there holds $\Xi(\mathcal{I}(G)) = 2\Delta(G)$, meaning that G is the disjoint union of complete bipartite graphs.

A fractional version was proved in [3]. The fractional covering number $\beta^*(H)$ of a hypergraph H is $\min\{\sum_{h \in H} \alpha_h \mid \sum_{h \in H} \alpha_h \chi(h) \geq \bar{1}_V\}$, where $\chi(h) \in \mathbb{R}^{V(H)}$ is the characteristic function of h.

Theorem 14.6. $\beta^*(\mathcal{I}(G) \cap \mathcal{M}) \leq \max(2\Delta(G), \Xi(\mathcal{M}))$.

Question 14.7.

(1) Why isn't it true that $\beta^*(\mathcal{I}(G) \cap \mathcal{M}) \leq \max(2\Delta(G) - 1, \Delta(\mathcal{M}))$.
(2) For the free matroid (in which every set is independent) the result is $\beta(\mathcal{I}(G) \cap \mathcal{M}) \leq \max(2\Delta(G), \Delta(\mathcal{M})) + 1$. Are there interest other ing matroids in which $\Delta(\mathcal{M})) + 1$ in the above can be replaced by a quantity smaller than $2\Delta(\mathcal{M})$?

14.1. **Rota's conjecture.** A famous conjecture of Rota [29] is:

Conjecture 14.8. *Let $\mathcal{P}$ be a partition matroid and $\mathcal{M}$ any matroid on the same ground set. Suppose that $\beta(\mathcal{P}) = \beta(\mathcal{M}) = n$ and*
(*) Each part of $\mathcal{P}$ belongs to $\mathcal{M}$.
Then $\beta(\mathcal{P} \cap \mathcal{M}) = n$.

The validity of this conjecture would mean that condition (*) enables removing the "+1" in Conjecture 14.4.

15. Hall's theorem for hypergraphs

For a hypergraph H we denote by $L(H)$ the line graph of H. It has $E(H)$ as its vertex set, and two edges (vertices of $L(H)$) are connected (by an edge of $L(H)$) if they intersect.

We can now prove Theorem 4.2. For the reader's convenience let us re-state it.

Theorem 15.1 ([13])**.** *Let $\mathcal{H} = (H_1, \ldots, H_m)$ be a family of r-uniform hypergraphs. If*

$$\nu(\mathcal{H}_I) > r \cdot (|I| - 1)$$

for all $I \subseteq [m]$, then $\mathcal{H}$ has a system of disjoint representatives.

Part (i) of Theorem 8.2 and the following observation imply that

$$\eta(\mathcal{M}(\mathcal{H}_I)) = \eta\Big(\mathcal{I}\big(L(\mathcal{H}_I)\big)\Big) \geq \gamma^i(L(\mathcal{H}_I)).$$

Therefore Theorem 15.1 is a direct corollary of Theorem 4.3.

Observation 15.2. *If H is an r-uniform hypergraph, then $\gamma^i(L(H)) \geq \frac{\nu(H)}{r}$.*

This is true because a matching in H is an independent set in $L(H)$, and in an r-uniform hypergraph an edge can meet at most r disjoint edges, so to dominate a matching of size $\nu(H)$, it needs at least $\frac{\nu(H)}{r}$ edges.

In [7] a stronger version was proved, in which the same conclusion is implied by a (usually much) weaker assumption, that $\nu^* \left(\bigcup \mathcal{H}_I\right) > r \cdot \left(\big|I\big| - 1\right)$ for all $I \subseteq [m]$. Here ν^* is the fractional matching number, namely the maximum sum of weights on the edges, that do not add up to more than 1 on any "star" (the edges containing a specific vertex). The methods used in the proof are beyond the scope of this survey.

16. Cooperative covers

To formulate sufficient conditions for the statement $\beta(\mathcal{C}) \leq k$ as in the proof of Theorem 14.2, we used the join of k copies of $\mathcal{C}$. Taking the join of k not necessarily identical complexes yields sufficient conditions for the existence of *cooperative covers*. Given complexes $\mathcal{C}_1, \dots, \mathcal{C}_k$ on the same vertex set V, a *cooperative cover* is a choice $\sigma_i \in \mathcal{C}_i$ for each $i \in [k]$, satisfying $\bigcup_{1 \leq i \leq k} \sigma_i = V$. The same proof as that of Theorem 14.2 gives the following.

Theorem 16.1. *If $\sum_{1 \leq i \leq k} \eta(\mathcal{C}_i[S]) \geq |S|$ for every $S \subseteq V$ then $\mathcal{C}_1, \dots, \mathcal{C}_k$ have a cooperative cover.*

16.1. Cooperative colorings of graphs.

A special case of cooperative covers is that of *cooperative colorings of graphs*. Given graphs $G_1, \dots, G_k$ on V, a cooperative coloring is a cooperative cover of the complexes $\mathcal{I}(G_1), \dots, \mathcal{I}(G_k)$.

By Theorem 8.3 if $\Delta(G_i) \leq d$, then $\eta(\mathcal{I}(G_i)[S]) \geq \frac{\tilde{\gamma}(G[S])}{2} \geq \frac{|S|}{2d}$, and hence by Theorem 16.1 any $2d$ graphs of maximal degree at most d have a cooperative coloring.

The fact that every graph G is $\Delta(G) + 1$ colorable suggests that $d + 1$ graphs of mximal degree d have a cooperative coloring. However, this is false. In [20] examples are given for every d, of $d + 1$ graphs of maximal degree d having no cooperative coloring.

A small result is in the following, whose main interest is that it seems difficult to prove without topology.

Theorem 16.2 ([14])**.** *Any three cycles on the same vertex set have a cooperative coloring.*

The proof uses a lemma of Meshulam that can be derived from his bounds Theorem 8.1.

Lemma 16.3. *For a cycle C of length k and a path P of length $k - 1$, we have:*

(1) $\eta(\mathcal{I}(C)) = nint(\frac{k}{3})$ ($nint(x)$ denotes the rounding of x to the closest integer.)

(2) $\eta(\mathcal{I}(P)) > \frac{k}{3}$.

Proof of Theorem 16.2. Let C_1, C_2, C_3 be the three cycles. We use the same construction as above - taking three copies W_1, W_2, W_3 of the vertex set V, where W_i consists of a copy u_i of every vertex u; putting on W_i the cycle C_i, letting $\mathcal{C} = *_{1\le i\le 3}\mathcal{I}(C_i)$ and partitioning the vertex set into sets U_v, $v \in V$, where $U_v = \{v_1, v_2, v_3\}$. We are looking for a $\mathcal{C}$-transversal $T : V \to \cup_{1\le i\le 3} W_i$ with respect to $(U_v)_{v\in V}$. Choose an arbitrary vertex v, and let $T(v) = v_1$. Remove v_1 from W_1, together with its two neighbors a, b. It suffices to show that in the remaining vertices there is an $\mathcal{I}(\mathcal{C})$-transversal for $(U_u)_{u\neq v}$. By Lemma 16.3 η of the union every k sets $U_v - \{a, b\}$ is at least $\frac{3k-2}{3}$, and since it is an integer, it is at least k, befitting the condition of Topological Hall.

□

17. Ryser's conjecture for 3-partite hypergraphs

A conjecture that prompted the research in this field is named after Ryser, though it appeared in the Ph.D thesis of Henderson [28], his student.

Conjecture 17.1. *In an r-partite r-uniform hypergraph $\tau \le (r-1)\nu$.*

Here τ is the minimal number of vertices meeting all edges. For $r = 2$ the conjecture is in fact a well-known theorem of König: in bipartite graphs the covering number is equal to the matching number. In [2] the case $r = 3$ of the conjecture was proved.

Theorem 17.2. *In a tripartite hypergraph $\tau \le 2\nu$.*

Outline of proof. Let H be a tripartite hypergraph with sides V_1, V_2, V_3. Every vertex in $v \in V_1$ can be viewed as a set of pairs, namely $\{(a, b) \in V_2 \times V_3 \mid (v, a, b) \in E(H)\}$. A matching in the hypergraph is then a partial choice function whose image is a matching. Let T be a subset of V_1 maximizing $|T| - \eta(\mathcal{M}[K[T]])$, where $K[T]$ is the set of edges in $V_2 \times V_3$ connected to T and $\mathcal{M}$ is the matching complex in $V_2 \times V_3$.

Since in a graph an edge dominates at most two edges from any given matching, $\gamma^i(G) \ge \frac{\nu(G)}{2}$, and hence by Theorem 8.2 $\eta(\mathcal{M}(G)) \ge \frac{\nu(G)}{2}$, which by König's theorem implies that if G is bipartite then $\tau(G) \le 2\eta(\mathcal{M}(G))$.

By Theorem 4.5

$$\nu(H) \ge |V_1| - def(V_1) = |V_1| - |T| + \eta(\mathcal{M}(K(T))). \tag{7}$$

By the above let Z covers $K(T)$ with $|Z| \le 2\eta(\mathcal{M}(K(T)))$. The set $Z \cup (V_1 \setminus T)$ is a cover for H. Hence $\tau(H) \le 2\eta + |V_1| - |T| \le 2(\eta + |V_1| - |T|) \le 2\nu(H)$. □

In [3] a matroidal version was proved.

Theorem 17.3. *Let $\mathcal{L}, \mathcal{M}, \mathcal{N}$ be three matroids on V. Then*

$$\min\Big(rank_{\mathcal{L}}(A) + rank_{\mathcal{M}}(B) + rank_{\mathcal{N}}(C)\Big) \le 2rank(\mathcal{L} \cap \mathcal{M} \cap \mathcal{N}),$$

where the minimum is taken over all partitions (A, B, C) of V.

18. Kőnig's theorem for point-trees bipartite hypergraphs

Let T be a tree and Z a set disjoint from $V(T)$. A *point-tree* is $e = \{z\} \cup t$, where $z \in Z$ and t is the vertex set of a sub-tree of T, and let $z(e) = z$ and $t(e) = t$. For a collection H of point-trees on $Z \cup V(T)$, let $s(H)$ be the minimum size of the set $C \subseteq Z \cup \{t(e) : e \in H\}$ satisfying that for every $h \in H$, either $z(h) \in C$ or $t(h) \in C$.

Theorem 18.1 ([8])**.** *For any collection H of point-trees, $s(H) \leq \nu(H)$.*

König's theorem is the case that all trees t are singletons.

A graph is called *chordal* if it contains no induced cycle of length greater than 3. In [25] it was proved that a graph is chordal if and only if it is the line graph of a collection of the vertex sets of sub-trees of a tree. The crux of the proof of Theorem 18.1 is the following result, whose proof can be found in [9].

Theorem 18.2. *In a chordal graph $\gamma^i = \gamma$.*

Recall that for a hypergraph J we denote by $L(J)$ the line graph of J.

Proof of Theorem Theorem 18.1. The proof goes along the same lines of the proof of Kőnig's theorem from Hall's theorem, namely using the deficiency version (Theorem 4.5). For a subset Y of Z, let $T(Y) = \{t(h) \mid h \in H, y \in h \cap Y\}$ and let

$$def_\gamma(Y) = |Y| - \gamma(L(T(Y))).$$

By Theorem 8.2,

$$\eta(\mathcal{I}(T(Y))) \geq \gamma^i(L(T(Y))) = \gamma(L(T(Y))).$$

The γ-deficiency of H, denoted by $def_\gamma(H)$, is $\max_{Y \subseteq Z} def_\gamma(Y)$. Assume this maximum is attained by $Y_0 \subseteq Z$. Applying deficiency version of Topological Hall (Theorem 4.5) yields $\nu(H) \geq |Z| - def_\gamma(H)$. On the other hand, the dominating set of $L(T(Y_0))$ together with $Z \setminus Y_0$ yields $s(H) \leq |Y_0| - def_\gamma(H) + |Z| - |Y_0|$, which completes the proof. □

19. The cycle + triangles conjecture

In [23] Du and Hsu offered the following conjecture, which was later proved in a stronger form by Fleischner and Stiebitz [24].

Conjecture 19.1 (The Du-Hsu conjecture, later Fleischner-Stiebitz theorem)**.** *Let G be a 4-regular graph that is the union (edge-wise) of one C_{3k} and k pair-wise vertex-disjoint triangles on the vertex set of the cycle. Then $\alpha(G) \geq k$.*

Here, as usual, $\alpha(G) := rank(\mathcal{I}(G))$. Fleischner and Stiebitz used a theorem of Alon and Tarsi to prove a stronger result showing that the union of the cycle and the triangles is 3-colorable (each of the three colors satisfying the condition of the Du-Hsu conjecture). Another elementary proof was given by Horst Sachs [40]. For many years the only solutions to the Du-Hsu conjecture went through 3-colorability. Topological Hall yields the conjecture directly.

Proof. Let the triangles be $T_1, \dots, T_k$. The desired set is an $\mathcal{I}(C_{3k})$-transversal for the system $\mathcal{V} = (V_1, \dots, V_k)$, where $V_i = V(T_i)$ for each $i \in [k]$. By Lemma 16.3, $\eta(\mathcal{I}(C_{3k})[A]) \geq \frac{|A|}{3}$ for any subset A of $V(C_{3k})$. Given any $I \subseteq [k]$, apply this fact to $\mathcal{V}_I$ to deduce $\eta(\mathcal{I}(C_{3k})[\mathcal{V}_I]) \geq \frac{3|I|}{3} = |I|$, which is the condition in Topological Hall. □

The present proof yields a little more.

Theorem 19.2. *Let G be the disjoint union of cycles, each of length $\equiv 0 \bmod (3)$ or $\equiv 2 \bmod (3)$, and let $\mathcal{P}$ be a system of triangles whose vertex set partitions $V(G)$. Then $\mathcal{P}$ has an $\mathcal{I}(G)$-transversal.*

20. Challenges

20.1. **Weighted versions.** Matching theorems obtained by combinatorial methods often have weighted generalizations. For example, if w is a system of weights put on the edges of a bipartite graph G, then $\nu_w(G) = \tau_w$, where $\nu_w = \max(\sum_{e \in F} w(e) \mid F \in \mathcal{M}(G)\}$ and $\tau_w = \min\{\sum_{v \in V(G)} g(v) \mid \sum_{v \in e} g(v) \geq w(e)$ for all $e \in E(G)\}$. The weighted $r = 3$ version of Ryser's conjecture would follow from the following conjecture:

Conjecture 20.1. *In an r-partite weighted hypergraph $\tau_w \leq (r-1)\nu_w$.*

A similar situation exists with respect to matchings and covers in weighted 2-intervals. There, too, in the non-weighted case it is known that $\tau \leq 2\nu$, the only proof known is topological, and the weighted case is not known (see [16] for a partial result).

20.2. **Partial rainbow sets.**

Conjecture 20.2. *If $\nu_R = k$ then there exists $I \subseteq [m]$ of size at most k with $\eta(\mathcal{V}_I) < |I|$ or $\eta(\bigcup \mathcal{V}) \leq k$.*

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