COLORING, LIST COLORING, AND FRACTIONAL COLORING IN INTERSECTIONS OF MATROIDS

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Abstract. It is known that in matroids the difference between the chromatic number and the fractional chromatic number is smaller than 1, and that the list chromatic number is equal to the chromatic number. We investigate the gap within these pairs of parameters for hypergraphs that are the intersection of a given number k of matroids. We prove that in such hypergraphs the list chromatic number is at most k times the chromatic number and at most $2k-1$ times the maximum chromatic number among the k matroids. We study the relationship between three polytopes associated with k-sets of matroids, and connect them to bounds on the fractional chromatic number of the intersection of the members of the k-set. This also connects to bounds on the matroidal matching and covering number of the intersection of the members of the k-set. The tools used are in part topological.

1. Preliminaries

1.1. **Hypergraphs and complexes.** A *hypergraph* $\mathcal{H}$ is a collection of subsets, called *edges*, of its *vertex set* (or *ground set*), a finite set $V = V(\mathcal{H})$. Throughout the paper we assume that there is no isolated vertex in $\mathcal{H}$, i.e., every vertex $v \in V(\mathcal{H})$ belongs to some edge of $\mathcal{H}$. If all edges of a hypergraph $\mathcal{H}$ are of the same size k we say that $\mathcal{H}$ is *k-uniform*, or that it is a *k-graph*. A hypergraph $\mathcal{H}$ is *k-partite* if its vertex set can be divided into k parts $V_1, \dots, V_k$ so that for every edge $S \in \mathcal{H}$, $|S \cap V_i| = 1$ for each $1 \le i \le k$. For $U \subseteq V$, $\mathcal{H}[U] = \{S \in \mathcal{H} \mid S \subseteq U\}$ is the *subhypergraph* of $\mathcal{H}$ *induced* on U.

A *matching* in a hypergraph is a set of disjoint edges, and a *cover* is a set of vertices meeting all its edges. The set of matchings in $\mathcal{H}$ is denoted by $\mathcal{M}(\mathcal{H})$. The maximum size of a matching in $\mathcal{H}$ is denoted by $\nu(\mathcal{H})$, and the minimum size of a cover by $\tau(\mathcal{H})$. Obviously, $\tau(\mathcal{H}) \ge \nu(\mathcal{H})$.

A set of vertices of $\mathcal{H}$ is said to be *independent* if it does not contain an edge. The complex of independent sets in $\mathcal{H}$ is denoted by $\mathcal{I}(\mathcal{H})$. Clearly, both $\mathcal{M}(\mathcal{H})$ and $\mathcal{I}(\mathcal{H})$ are non-empty (both include the empty set) and closed under taking subsets. Hypergraphs satisfying these two conditions are called *(abstract simplicial) complexes*, and their edges are also called *faces*. This terminology is borrowed from topology.

We assume (except for one explicit deviation — Definition 1.1 of the "join" below) that all complexes have the same set, denoted by V, as their ground set.

For a complex $\mathcal{C}$ and $U \subseteq V$, let $rank_{\mathcal{C}}(U) = \max_{S \in \mathcal{C}[U]} |S|$. The *rank* of $\mathcal{C}$, denoted by $rank(\mathcal{C})$, is $rank_{\mathcal{C}}(V)$.

A complex $\mathcal{C}$ is said to be a *flag complex* if it is 2-determined, meaning that $e \in \mathcal{C}$ whenever $\binom{e}{2} \subseteq \mathcal{C}$. (Note that $\binom{S}{m} = \{T \subseteq S \mid |T| = m\}$.)

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Definition 1.1. The *join* $\mathcal{C} * \mathcal{D}$ of two complexes on disjoint ground sets is $\{A \cup B \mid A \in \mathcal{C}, B \in \mathcal{D}\}$. If $V(\mathcal{C}) \cap V(\mathcal{D}) \neq \emptyset$, we define $\mathcal{C} * \mathcal{D}$ by first making a copy of $\mathcal{D}$ on a ground set that is disjoint from that of $\mathcal{C}$, and then taking the join of $\mathcal{C}$ and the copy of $\mathcal{D}$.

1.2. **Colorings, list colorings, and fractional colorings.** Given a complex $\mathcal{C}$, a *coloring* by $\mathcal{C}$ is a set of faces of $\mathcal{C}$ whose union is the ground set V. The *chromatic number* $\chi(\mathcal{C})$ of $\mathcal{C}$ is the minimum size (number of faces) of a coloring.

Let L_v be a set of *permissible colors* at every $v \in V$. A *list coloring* with respect to these is a function $f : V \to \cup_{v \in V} L_v$ satisfying $f(v) \in L_v$ for every $v \in V$. It is said to be $\mathcal{C}$-*respecting* if $f^{-1}(c) \in \mathcal{C}$ for every color $c \in \cup_{v \in V} L_v$. The list chromatic number $\chi_\ell(\mathcal{C})$ is the minimum number p such that any system of lists L_v of size p has a $\mathcal{C}$-respecting list coloring. Obviously,

$$\chi_\ell(\mathcal{C}) \geq \chi(\mathcal{C}).$$

The second main concept we shall study is that of fractional colorings.

Definition 1.2. Given a complex $\mathcal{C}$ on V and $\vec{w} \in \mathbb{R}^V_{\geq 0}$, a *$\vec{w}$-fractional coloring* of $\mathcal{C}$ is a function $f : \mathcal{C} \to \mathbb{R}_{\geq 0}$ satisfying $\sum_{S \in \mathcal{C}: v \in S} f(S) \geq w(v)$ for every $v \in V$. The *fractional $\vec{w}$-chromatic number*, denoted by $\chi^*(\mathcal{C}, \vec{w})$, is the minimum of $\sum_{S \in \mathcal{C}} f(S)$ over all $\vec{w}$-fractional colorings f of $\mathcal{C}$. A $\vec{1}$-fractional coloring is plainly called a *fractional coloring*, and $\chi^*(\mathcal{C}, \vec{1})$ is denoted by $\chi^*(\mathcal{C})$.

1.3. **Matroids.** A complex $\mathcal{M}$ is called a *matroid* if for all $S, T \in \mathcal{M}$ satisfying $|S| < |T|$ there exists $v \in T \setminus S$ such that $S \cup \{v\} \in \mathcal{M}$. The edges of a matroid are said to be *independent*. A *base* of a matroid is a maximal independent set. A *circuit* is a minimal dependent set (the name comes from graphic matroids, namely matroids consisting of acyclic sets of edges in a graph). This is compatible with the terminology of independent sets in hypergraphs, once we note that $\mathcal{M} = \mathcal{I}(\mathcal{H})$ where $\mathcal{H}$ is the set of circuits.

For $A \subseteq V(\mathcal{M})$, let

$$span_{\mathcal{M}}(A) = A \cup \Big\{x \in V \mid \{x\} \cup S \notin \mathcal{M} \text{ for some } S \in \mathcal{M}[A]\Big\}.$$

$A \subseteq V$ is called *spanning* if $span_{\mathcal{M}}(A) = V$.

Throughout the paper we assume that all matroids are loopless, namely all singletons are independent.

Given a partition $\mathcal{P} = (P_1, P_2, \ldots, P_m)$ of V, let $\mathcal{T}(\mathcal{P})$ be the set of all subsets of V meeting each P_i for $1 \leq i \leq m$ in at most one vertex. $\mathcal{T}(\mathcal{P})$ is called a *partition matroid*.

In a seminal paper [11], Edmonds showed how combinatorial duality (min-max results) can sometimes be formulated in terms of the intersection of two matroids. The classical case is the König–Hall theorem, which can be viewed as a statement on the intersection of two partition matroids, defined on the edge set of a bipartite graph. The parts in each are the stars in one of the sides. A matching in this graph is a set of edges belonging to the intersection of the two matroids, and a matching of one side is a base in one matroid that is independent in the other.

Intersections of more than two matroids are more complex: $\min - \max$ results are no longer available, algorithms for finding maximum size sets in the intersection are no longer polynomial, and necessary and sufficient conditions for the existence of certain objects are replaced by sufficient conditions.

Let $\mathcal{D}^k$ be the collection of sets $\mathcal{L} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\}$ of matroids. Let $MINT_k = \{\bigcap \mathcal{L} \mid \mathcal{L} \in \mathcal{D}^k\}$, namely the set of complexes that are intersections of k matroids on the same ground set.

The topic of the paper is properties of complexes belonging to $MINT_k$. Though possibly familiar, it is worthwhile mentioning the primary facts about such objects. The first, that was proved more than once, is that every complex is in some $MINT_k$.

Theorem 1.3. [21, 12, 25] *Every complex is the intersection of matroids.*

Proof. For $e \subseteq V$, let $\mathcal{M}_e = \{S \subseteq V \mid S \not\supseteq e\}$. $\mathcal{M}_e$ is easily seen to be a matroid, and $\mathcal{C} = \bigcap_{e \notin \mathcal{C}} \mathcal{M}_e$. □

In his M.Sc thesis [20], András Imolay addressed the question of how many matroids are needed in the intersection. Let $\kappa(n)$ be the maximum, over all complexes $\mathcal{C}$ on n vertices, of the minimum number of matroids whose intersection is $\mathcal{C}$.

Theorem 1.4 (Theorem 6.7 in [20])**.** $\binom{n-1}{\lfloor (n-1)/2 \rfloor} \leq \kappa(n) \leq \binom{n}{\lfloor n/2 \rfloor}$.

A special role will be played by systems of partition matroids. Given a k-partite hypergraph $\mathcal{H}$, we construct a collection $\mathcal{L} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\}$ of k partition matroids, each having $E(\mathcal{H})$ as ground set, where each parts of $\mathcal{M}_i$ is a star S_x, namely the set of edges incident to a vertex x in the ith side of $\mathcal{H}$. This system of matroids is denoted by $\mathcal{L}(\mathcal{H})$.

Conversely, let $\mathcal{L} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k\}$ be a system of partition matroids, defined by the partitions $\mathcal{P}^i = (P^i_1, P^i_2, \ldots, P^i_{m_i})$ of the ground set V for $i \in [k]$. Then the complex $\cap_{i=1}^k \mathcal{M}_i$ is the matching complex of a k-partite k-uniform hypergraph $\mathcal{K} = \mathcal{K}(\mathcal{L})$, whose vertices are the parts P^i_j, and each of its sides S_i is $\{P^i_j \mid 1 \leq j \leq m_i\}$. Every vertex $v \in V$ corresponds to an edge $e(v) = \{P^i_j \mid 1 \leq i \leq k, v \in P^i_j\}$ of $\mathcal{K}$. Then $\cap_{i=1}^k \mathcal{M}_i = \mathcal{M}(\mathcal{K})$, the set of matchings in $\mathcal{K}$. Clearly,

$$(1) \qquad \mathcal{L}(\mathcal{K}(\mathcal{L})) = \mathcal{L} \quad \text{and} \quad \mathcal{K}(\mathcal{L}(\mathcal{K})) = \mathcal{K}.$$

These constructions are useful in characterizing intersections of partition matroids.

Theorem 1.5. *The following conditions are equivalent:*

(i) $\mathcal{C} = \mathcal{I}(G)$ *for some graph* G*,*
(ii) $\mathcal{C}$ *is a flag complex,*
(iii) $\mathcal{C}$ *is the intersection of* k *partition matroids for some* k*,*
(iv) $\mathcal{C}$ *is the matching complex of a* k*-partite hypergraph for some* k*.*

Moreover, (iii) *and* (iv) *are equivalent for each* k *separately.*

Proof. (i) ⇒ (ii) is true by the definition of an independent set as a set in which each pair of elements is independent. To prove (ii) ⇒ (i) (this follows from the other implications, but uses an observation worth noting) note that if $\mathcal{C}$ is a flag complex then $\mathcal{C} = \mathcal{I}(G)$ for the graph whose edge set is $\{xy \mid xy \notin \mathcal{C}\}$. (iv) ⇒ (i) is true since the matching complex $\mathcal{M}(H)$ of a hypergraph H is $\mathcal{I}(L(H))$, where $L(H)$ is the line graph of the hypergraph H. (iii) ⇔ (iv) follows from (1). To prove (ii) ⇒ (iii) note that if $|e| = 2$ then the matroid $\mathcal{M}_e$ in the proof of Theorem 1.3 is a partition matroid, with parts e, and all singletons disjoint from e. □

1.4. **Polytopes.** Another viewpoint on the intersection of matroids, introdcued by Edmonds, is that of polytopes, which are particularly useful in studying fractional colorings.

For a subset A of V let $\mathbf{1}_A \in \mathbb{R}^V$ be the *characteristic function* of A, namely the function taking value 1 on elements of A and 0 elsewhere. Functions will also be viewed as vectors, so a real-valued function f on a set V is also denoted by $\vec{f} \in \mathbb{R}^V$. We write $f[V]$ for $\sum_{v\in V} f(v)$.

The polytope $P(\mathcal{C})$ of a complex $\mathcal{C}$ on V is the convex hull in $\mathbb{R}^V_{\geq 0}$ of the characteristic vectors of the edges of $\mathcal{C}$. A polytope $Z \subseteq \mathbb{R}^V_{\geq 0}$ is *closed down* if $z \in Z$ and $0 \leq y \leq z$ imply $y \in Z$.

Observation 1.6. *$P(\mathcal{C})$ is closed down.*

Proof. Let

$$\vec{v} = \sum_{S\in\mathcal{C}} \lambda_S \mathbf{1}_S \in P(\mathcal{C}), \tag{2}$$

where $\lambda_S \geq 0$ and $\sum_{S\in\mathcal{C}} \lambda_S = 1$. Let $\vec{0} \leq \vec{u} \leq \vec{v}$. We claim that $\vec{u} \in P(\mathcal{C})$. Using induction, it suffices to consider the case that $\vec{u}$ and $\vec{v}$ differ just in one coordinate, namely $\vec{u} = \vec{v} - \alpha e_x$ for some $x \in V$. Furthermore, by the convexity of $P(\mathcal{C})$, it is enough to prove the case $\vec{u}(x) = 0$. Since $\mathcal{C}$ is a complex, we can replace all S's containing x in (2) by $S \setminus \{x\}$ to obtain $\vec{u}$. □

1.5. **Matroids intersection vs. matroidal cooperative covers.** Given a k-set of matroids $\mathcal{L} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\} \in \mathcal{D}^k$, let $\bar{\nu}(\mathcal{L}) = rank(\bigcap \mathcal{L})$ and $\bar{\tau}(\mathcal{L}) = \min\{\sum_{i=1}^k rank_{\mathcal{M}_i}(V_i) \mid \cup_{i=1}^k V_i = V\}$. A set belonging to $\bigcap \mathcal{L}$ is called a *matoridal matching* of $\mathcal{L}$, and a k-tuple of functions $(\mathbf{1}_{U_1}, \ldots, \mathbf{1}_{U_k})$ for which $\cup_{i=1}^k span_{\mathcal{M}_i}(U_i) = V$ is called a *matroidal cooperative cover*. This is the reason for the notation $\bar{\nu}$ and $\bar{\tau}$ (see Section 10 for more details). To see the relation between $\bar{\tau}(\mathcal{L})$ and the matroidal cooperative cover, taking $U_i \in \mathcal{M}_i$ and setting $V_i = span_{\mathcal{M}_i}(U_i)$ for each $i \in [k]$, we have

$$\sum_{i=1}^k \mathbf{1}_{U_i}[V] = \sum_{i=1}^k rank_{\mathcal{M}_i}(V_i).$$

Given a maximum size set X in $\bigcap \mathcal{L}$ and any representation of V as $\cup_{i=1}^k V_i$ we have

$$|X| = |X \cap (\cup_{i=1}^k) V_i| \leq \sum_{i=1}^k |X \cap V_i| \leq \sum_{i=1}^k rank_{\mathcal{M}_i}(V_i).$$

This shows that

$$\bar{\nu}(\mathcal{L}) \leq \bar{\tau}(\mathcal{L}). \tag{3}$$

Edmonds' two-matroids intersection theorem is that for $k = 2$ equality holds.

Theorem 1.7 ([11])**.** *For any $\mathcal{L} \in \mathcal{D}^2$, we have $\bar{\tau}(\mathcal{L}) = \bar{\nu}(\mathcal{L})$.*

We shall later prove that equality holds also for fractional versions of $\bar{\nu}$ and $\bar{\tau}$, for every k. It is not hard to prove that $\bar{\tau}(\mathcal{L}) \leq k\bar{\nu}(\mathcal{L})$ — we shall return to variants of this inequality, and also to the basic conjecture in the field, of which Edmonds' theorem is a special case.

Conjecture 1.8 (Conjecture 7.1 in [2])**.** $\bar{\tau}(\mathcal{L}) \leq (k-1)\bar{\nu}(\mathcal{L})$.

[2, Theorem 7.3] yields the conjecture for $k = 3$.

If $\mathcal{L} \in \mathcal{D}^k$ is a collection of partition matroids and $H = \mathcal{K}(\mathcal{L})$, which is a k-partite hypergraph, then $\bar{\nu}(\mathcal{L}) = \nu(H)$ and $\bar{\tau}(\mathcal{L}) = \tau(H)$ (Observation 10.24). Thus a conjecture of Ryser, stating that in k-partite hypergraphs $\tau \leq (k-1)\nu$, is a special case of Conjecture 1.8.

2. Preview

The unifying theme of the paper is that belonging to $MINT_k$ entails quantifiably "tame" behavior of the complex. This is manifested in two ways. One, studied in Sections 3–7, is that the list chromatic number is not far from the ordinary chromatic number. This is proved via an observation, that a topological tool used to bound the chromatic number works just as well for the list chromatic number. This leads to two results. One (Theorem 5.3) is that if $\mathcal{C} \in MINT_k$ then

$$\chi_\ell(\mathcal{C}) \leq k\chi(\mathcal{C}).$$

This solves a conjecture posed by Király [23] and also by Bérczi, Schwarcz, and Yamaguchi [7]. The other result (Corollary 7.2) is that if $\mathcal{C} = \cap_{i=1}^k \mathcal{M}_i \in MINT_k$, then

$$\chi_\ell(\mathcal{C}) \leq (2k-1) \max_{1 \leq i \leq k} \chi(\mathcal{M}_i).$$

The proof of the latter requires new topological tools. If all matroids $\mathcal{M}_i$ are partition matroids then the $2k-1$ factor can be replaced by k (Corollary 6.3), i.e.,

$$\chi_\ell(\mathcal{C}) \leq k \max_{1 \leq i \leq k} \chi(\mathcal{M}_i).$$

The latter is probably true (and also not sharp) for general matroids.

Sections 8 and on deal with the tameness of intersections of matroids with respect to fractional colorings. The motivation comes from two theorems of Edmonds on the intersection of two matroids. One is Theorem 1.7. The other, closely related, is that for any two matroids, $\mathcal{M}$ and $\mathcal{N}$, the following holds.

Theorem 2.1 ([10]). *$P(\mathcal{M} \cap \mathcal{N}) = P(\mathcal{M}) \cap P(\mathcal{N})$.*

An obvious challenge is to extend this theorem to any number of matroids. In Section 10 we define two notions: matchings and cooperative covers, as well as their fractional and weighted versions, for k-sets of matroids. We prove (Theorem 10.4) that the fractional weighted cooperative covering number equals the fractional weighted matching number for any k-set of matroids.

Following the footsteps of a result of Füredi [14] and its weighted version proved by Füredi, Kahn and Seymour [15], we consider a matroidal fractional weighted Ryser-type conjecture (Conjecture 10.11). We prove its equivalence to the following conjecture on polytopes.

Conjecture 2.2. *For $\mathcal{C} = \cap_{i=1}^k \mathcal{M}_i \in MINT_k$,*

$$(k-1)P(\mathcal{C}) \supseteq \cap_{i=1}^k P(\mathcal{M}_i).$$

Theorem 2.1 is the case $k = 2$.

In Section 8 we show an equivalence to yet another conjecture.

Conjecture 2.3. *For $\mathcal{C} = \cap_{i=1}^k \mathcal{M}_i \in MINT_k$ and $\vec{w} \in \mathbb{R}^V_{\geq 0}$,*

$$\chi^*(\mathcal{C}, \vec{w}) \leq (k-1) \max_{1 \leq i \leq k} \chi^*(\mathcal{M}_i, \vec{w}).$$

Using known results, we prove these conjectures for $k \leq 3$ and for partition matroids (see Section 10.2). For general k and general matroids, we prove these conjectures with k replacing $k-1$ (Corollaries 10.14–10.15).

Section 11 is devoted to the study of the fractional $\vec{w}$-chromatic number $\chi(\mathcal{C}, \vec{w})$ for general complexes $\mathcal{C}$.

The last section is devoted to a brief discussion of the combination of the two main themes — a fractional version of list coloring.

3. A topological tool

3.1. **$\mathcal{C}$-transversals.** Hall's theorem is about the existence of injective choice functions. A more general notion is that of $\mathcal{C}$-transversals. Given a complex $\mathcal{C}$ and subsets S_i of $V(\mathcal{C})$ for $i \in I$, a *$\mathcal{C}$-transversal* is a choice function $x_i \in S_i$ for $i \in I$, whose image $\{x_i \mid i \in I\}$ is a face of $\mathcal{C}$.

This notion is particularly useful in two contexts.

(1) *Rainbow matchings.* A choice function whose domain is sets of hypergraph edges, and its image is a matching, is called a *rainbow matching.* A special case is matchings in *bipartite hypergraphs.* A hypergraph $\mathcal{B}$ is said to be bipartite if there is $A \subseteq V(\mathcal{B})$ meeting each edge at precisely one vertex: each $a \in A$ can be considered as a set S_a of edges, those that complement a to an edge of $\mathcal{B}$. A rainbow matching for this system is plainly a matching in $\mathcal{B}$.

(2) *Colorings.* A classic construction transforms any k-coloring of $\mathcal{C}$ (namely a set of k edges whose union is $V(\mathcal{C})$), to $\mathcal{C}$-transversals. Take the join $\mathcal{D} = *_{j \in [k]} \mathcal{C}$ of k copies of $\mathcal{C}$, and for every $v \in V$ let S_v be the set of copies of v. Then a $\mathcal{D}$-transversal T is a cover of V by k faces of $\mathcal{C}$ — the ith face being the image of T in the ith copy of $\mathcal{C}$. This is what we call "k-coloring".

Remark 3.1. The earliest reference we know to this construction is in Welsh's 1976 book, "Matroid theory" [33], where it is used to prove Edmonds' two matroids intersection theorem. But it may well be older than that.

The study of $\mathcal{C}$-transversals requires a more general tool than Hall's theorem. If $\mathcal{C}$ is a matroid, then such a tool is given by a theorem of Rado [30]. For general complexes, a topological tool has been developed. The basic theorem in this direction is that a $\mathcal{C}$-transversal exists if (but not only if) the union of any k sets S_i induces a complex of connectivity at least k (replacing "being of size at least k" in Hall's theorem). We next define this notion.

3.2. **Connectivity.** We denote by S^k the k-dimensional sphere, and by B^k is the k-dimensional ball. So, S^k is the boundary of B^{k+1}. B^0 is a single point, and accordingly $S^{-1} = \emptyset$.

A topological space X is *homotopically k-connected* if for every $-1 \leq i \leq k$, every continuous function from the i-dimensional sphere S^i to X can be extended to a continuous function from the ball B^{i+1} to X. Especially we define the empty space to be *homotopically (-2)-connected.* If $X = \emptyset$, then it is *homologically* (-1)-*connected.* If $X \neq \emptyset$, it is *homologically k-connected* if $\tilde{H}_i(X) = 0$ for all $0 \leq i \leq k$, where $\tilde{H}_i(X)$ is the reduced ith homology group of X. (See, e.g., Chapter 2 of [18] for the definition of $\tilde{H}_i(X)$. While the readers who are not familiar with this definition can skip it at this moment.)

Let $\eta(X)$ (resp. $\eta_H(X)$) be the maximum k for which X is homotopically k-connected (resp. homologically k-connected), plus 2. We shall generally not distinguish between the two notions, one reason being a result of Hurewicz.

Theorem 3.2 (Hurewicz, [18]). *We have $\eta_H \geq \eta$. Furthermore, if $\eta(X) \geq 3$, which signifies "X is simply connected", then $\eta(X) = \eta_H(X)$.*

An abstract complex $\mathcal{C}$ has a geometric realization $||\mathcal{C}||$, obtained by positioning its vertices in general position in $\mathbb{R}^n$ for large enough n ($n = 2 \cdot rank(\mathcal{C}) - 1$ suffices, for example any rank-2 complex, namely a graph, can be realized without fortuitous intersections in $\mathbb{R}^3$) and representing every face by the convex hull of its vertices, the "general position" condition guaranteeing that there are no unwanted intersections. We write $\eta(\mathcal{C})$ for $\eta(||\mathcal{C}||)$, and $\eta_H(\mathcal{C})$ for $\eta_H(||\mathcal{C}||)$.

Intuitively, $\eta(X)$ is the minimum dimension of a hole in X. For example, $\eta(S^n) = n+1$ because the hole, namely the non-filled ball, is of dimension $n+1$. For the disk B^n, $\eta(B^n) = \infty$, because there is no hole, meaning that images of S^n can be filled for every n. $\eta(X) = 0$ means $X = \emptyset$, and $\eta(X) = 1$ means that X is non-empty and is not path-connected. For another example, $\eta(\mathcal{C}) \geq 2$ means path-connectedness. The "2" is the size of the simplices that are used for filling those holes that are fillable. This is valid also for higher dimensions, which makes η a natural parameter in combinatorial settings.

3.3. **Lower bounds on η.** To apply Theorem 3.13 below, one needs combinatorially formulated lower bounds on $\eta(\mathcal{C})$. The following are the bounds used in this paper. See [2] for a reference.

Theorem 3.3. *$\eta_H(\mathcal{C} * \mathcal{D}) = \eta_H(\mathcal{C}) + \eta_H(\mathcal{D})$.*

Homotopic η does not yield equality here, but an inequality.

Theorem 3.4. *$\eta(\mathcal{C} * \mathcal{D}) \geq \eta(\mathcal{C}) + \eta(\mathcal{D})$.*

The connection of η to matroids is given by a result of Whitney [34].

Theorem 3.5. *If $\mathcal{M}$ is a matroid then $\eta(\mathcal{M}) \geq rank(\mathcal{M})$, with equality holding unless $\mathcal{M}$ has a co-loop, namely an element belonging to all bases, in which case $\eta(\mathcal{M}) = \infty$.*

The last statement is true since the matroid can be contracted to the co-loop. A generalization proved in [2] will play a central role in our investigation.

Theorem 3.6. *If $\mathcal{C} \in MINT_k$, then $\eta(\mathcal{C}) \geq \frac{rank(\mathcal{C})}{k}$.*

Example 3.7 (witnessing sharpness of the inequality). Let A be a set of size k, and let b be an element not belonging to A. Let $\mathcal{C} = \mathcal{P}(A) \cup \{\{b\}\}$. For every $a \in A$ let $\mathcal{M}_a$ be the matroid whose bases are A and $(A \setminus \{a\}) \cup \{b\}$. Then $\mathcal{C} = \bigcap_{a \in A} \mathcal{M}_a$, so $\mathcal{C} \in MINT_k$. We have $rank(\mathcal{C}) = k$ and $\eta(\mathcal{C}) = 1$, since $\mathcal{C}$ is not connected (there is no face in $\mathcal{C}$ of size two containing b).

For a graph G let $i\gamma(G)$ be the maximum over all sets $I \in \mathcal{I}(G)$ of the number of vertices needed to dominate I. The first combinatorial bound obtained on η was implicit in [5].

Theorem 3.8. *$\eta(\mathcal{I}(G)) \geq i\gamma(G)$.*

The next result follows from Theorem 3.6 and Theorem 1.5.

Corollary 3.9. *If $\mathcal{H}$ is k-uniform then $\eta(\mathcal{M}(\mathcal{H})) \geq \nu(\mathcal{H})/k$.*

In [3] this was strengthened.

Theorem 3.10 (Aharoni–Berger–Meshulam [3])**.** *For a k-uniform hypergraph $\mathcal{H}$,*

$$\eta(\mathcal{M}(\mathcal{H})) \geq \frac{\nu^*(\mathcal{H})}{k}.$$

Here $\nu^*(\mathcal{H})$ is the fractional matching number of $\mathcal{H}$ (see Definition 10.17).

A classic tool for obtaining lower bounds on η is an exact sequence of complexes, known as the "Mayer–Vietoris sequence". One of the consequences of the exactness is reminiscent of unimodularity. We do not know whether there is an intrinsic connection.

Lemma 3.11 (e.g., Section 2.2 of [18])**.** *For any pair $\mathcal{A}, \mathcal{B}$ of complexes,*

$$\eta_H(\mathcal{A}) \geq \min(\eta_H(\mathcal{A} \cup \mathcal{B}), \eta_H(\mathcal{A} \cap \mathcal{B})).$$

A homotopic version was proved in [2].

Meshulam [27] showed how Theorem 3.8 can be derived from this inequality. Another bound

$$\eta_H(\mathcal{I}(G)) \geq \gamma^E(G)$$

follows as a corollary, where $\gamma^E(G)$ is the minimum size of a set of edges whose union dominates G. Here, too, a homotopic version was proved in [4].

Theorem 3.12. *$\eta(\mathcal{I}(G)) \geq \gamma^E(G)$.*

In Section 7.2 we prove an extension to independence complexes of hypergraphs (see Theorem 7.7).

3.4. **Topological Hall.**

Theorem 3.13. *Let $\mathcal{C}$ be a complex, and let V_i for $1 \leq i \leq m$ be subsets of $V(\mathcal{C})$. If $\eta(\mathcal{C}[\bigcup_{i \in I} V_i]) \geq |I|$ for every $I \subseteq [m]$ then the sets V_i have a $\mathcal{C}$-transversal.*

This was proved implicitly in [5] and formulated explicitly by the first author (see remark following Theorem 1.3 in [27]). The special case when $\mathcal{C}$ is a matroid is Rado's theorem [30], mentioned above, in which η is replaced by its (almost) equal parameter, the rank (see Theorem 3.5).

Meshulam [27] proved the homological version of this theorem, which by Theorem 3.2 is stronger.

4. Definitions and known results of expansion and colorability

We adopt the conventions that $\lceil \frac{c}{\infty} \rceil = 1$ and $\frac{c}{0} = \infty$ whenever $c > 0$.

Definition 4.1. The *rank expansion number* $\Delta_r(\mathcal{C})$ of a complex $\mathcal{C}$ is

$$\max_{\emptyset \neq S \subseteq V} \frac{|S|}{rank_{\mathcal{C}}(S)}.$$

Since covering a set S by edges of size at most $rank_{\mathcal{C}}(S)$ requires at least $\frac{|S|}{rank_{\mathcal{C}}(S)}$ edges, we have

$$\chi(\mathcal{C}) \geq \Delta_r(\mathcal{C}). \quad (4)$$

Δ_r has a topological counterpart.

Definition 4.2. The *topological expansion number* $\Delta_\eta(\mathcal{C})$ of a complex $\mathcal{C}$ is

$$\max_{\emptyset \neq S \subseteq V} \frac{|S|}{\eta(\mathcal{C}[S])}.$$

For a complex $\mathcal{C}$, we define

$$\bar{\eta}(\mathcal{C}) = \min(\eta(\mathcal{C}), rank(\mathcal{C})).$$

The advantage of $\bar{\eta}$ over η is that it is always finite.

Definition 4.3. The *expansion number* $\Delta(\mathcal{C})$ of a complex $\mathcal{C}$ is

$$\max_{\emptyset \neq S \subseteq V} \frac{|S|}{\bar{\eta}(\mathcal{C}(S))}.$$

Then

$$\Delta_r(\mathcal{C}) \leq \Delta(\mathcal{C}) \quad \text{and} \quad \Delta_\eta(\mathcal{C}) \leq \Delta(\mathcal{C}). \tag{5}$$

The following was proved by Edmonds [9], extending a theorem of Nash-Williams [28] on graph arboricity.

Theorem 4.4. *In any matroid $\mathcal{M}$ we have $\chi(\mathcal{M}) = \lceil \Delta_r(\mathcal{M}) \rceil$.*

For general complexes an inequality holds.

Theorem 4.5 (Corollary 8.6 in [2])**.** $\chi(\mathcal{C}) \leq \lceil \Delta(\mathcal{C}) \rceil$.

5. List coloring

Recall that given a complex $\mathcal{C}$ and a list L_v of permissible colors at each $v \in V$, a $\mathcal{C}$-respecting list coloring is a choice (from the lists) function $f : V \to \cup_{v \in V} L_v$ such that $f^{-1}(c) \in \mathcal{C}$ for every color $c \in \cup_{v \in V} L_v$.

Virtually the same proof as that of Theorem 4.5 yields the following stronger result.

Theorem 5.1. *For any complex $\mathcal{C}$,*

$$\chi_\ell(\mathcal{C}) \leq \lceil \Delta_\eta(\mathcal{C}) \rceil.$$

Proof. For any system of lists $(L_v)_{v \in V(\mathcal{C})}$ satisfying $|L_v| = \lceil \Delta_\eta(\mathcal{C}) \rceil$, we shall find a $\mathcal{C}$-respecting list coloring. Let $J = \bigcup_{v \in V} L_v$ be the set of all the colors. For every color $j \in J$, let $F_j = \{v \in V(\mathcal{C}) \mid j \in L_v\}$ be the set of all elements such that the color j is in their lists. Form $W = \bigcup_{j \in J} \{j\} \times F_j$, which is the set of $|L_v|$ copies of each $v \in V(\mathcal{C})$. For each $j \in J$ let $\mathcal{C}_j$ be the complex $\{\{j\} \times \sigma \mid \sigma \in \mathcal{C}[F_j]\}$, namely a copy of $\mathcal{C}[F_j]$. Let $\mathcal{D} = *_{j \in J} \mathcal{C}_j$, namely the join of all $\mathcal{C}_j$ for $j \in J$, which is a complex on W.

For $v \in V(\mathcal{C})$, let $W_v = \{(j, v) \mid j \in L_v\}$ be the set of all the copies of v in W. For $I \subseteq V$, we claim that

$$\eta(\mathcal{D}[\cup_{v \in I} W_v]) \geq |I|. \tag{6}$$

To prove the claim, as $\cup_{v \in I} W_v = \cup_{j \in J} \{j\} \times (F_j \cap I)$, by Theorem 3.4 we have

$$\begin{aligned} \eta(\mathcal{D}[\cup_{v \in I} W_v]) &= \eta\Big((*_{j \in J} \mathcal{C}_j)[\cup_{j \in J} \{j\} \times (F_j \cap I)]\Big) \\ &\geq \sum_{j \in J} \eta(\mathcal{C}_j[\{j\} \times (F_j \cap I)]) = \sum_{j \in J} \eta(\mathcal{C}[F_j \cap I]). \end{aligned}$$

By the definition of $\Delta_\eta(\mathcal{C})$, we have

$$\eta(\mathcal{C}[F_j \cap I]) \geq \frac{|F_j \cap I|}{\lceil \Delta_\eta(\mathcal{C}) \rceil}.$$

Every vertex v of I appears in $|L_v| = \lceil \Delta_\eta(\mathcal{C}) \rceil$ many F_j, hence $\sum_{j\in J} |F_j \cap I| = \lceil \Delta_\eta(\mathcal{C}) \rceil \cdot |I|$. Hence

$$\eta(\mathcal{D}[\cup_{v\in I} W_v]) \geq \sum_{j\in J} \eta(\mathcal{C}[F_j \cap I]) \geq \frac{\sum_{j\in J} |F_j \cap I|}{\lceil \Delta_\eta(\mathcal{C}) \rceil} = \frac{\lceil \Delta_\eta(\mathcal{C}) \rceil \cdot |I|}{\lceil \Delta_\eta(\mathcal{C}) \rceil} = |I|,$$

completing the proof of (6).

By Theorem 3.13, there exists a choice function $\phi : V \to \cup_{v\in V} W_v$ such that $\phi(v) \in W_v$ for each $v \in V$ and $Im(\phi) \in \mathcal{D}$. Especially, $Im(\phi)$ is of the form $\bigsqcup_{j\in J} \{j\} \times \sigma_j$ for $\sigma_j \in \mathcal{C}[F_j] \subseteq \mathcal{C}$. Coloring every vertex $v \in V$ by the color $j = \phi(v)$ (so that $v \in \sigma_j$) produces then the desired $\mathcal{C}$-respecting list coloring. □

Combining with Theorem 3.5 and Theorem 4.4, this yields the following result.

Theorem 5.2. *For a matroid* $\mathcal{M}$, $\chi_\ell(\mathcal{M}) = \chi(\mathcal{M})$.

As noted above, this was first proved by Seymour [32].

In [7, 23] it was conjectured that there exists a constant α such that $\chi_\ell(\mathcal{C}) \leq \alpha\chi(\mathcal{C})$ for every $\mathcal{C} \in MINT_2$. Indeed, this is the case.

Theorem 5.3. *If* $\mathcal{C} \in MINT_k$ *then* $\chi_\ell(\mathcal{C}) \leq k\chi(\mathcal{C})$.

Proof. Assume that $S \subseteq V(\mathcal{C})$ attains the maximum in the definition of $\Delta_\eta(\mathcal{C})$, namely $\frac{|S|}{\eta(\mathcal{C}[S])} = \Delta_\eta(\mathcal{C})$. By Theorem 3.6, $rank_{\mathcal{C}}(S) \leq k\eta(\mathcal{C}[S])$. Together with (4), this yields

$$k\chi(\mathcal{C}) \geq k\chi(\mathcal{C}[S]) \geq \left\lceil k\frac{|S|}{rank_{\mathcal{C}}(S)} \right\rceil \geq \left\lceil \frac{|S|}{\eta(\mathcal{C}[S])} \right\rceil = \lceil \Delta_\eta(\mathcal{C}) \rceil \geq \chi_\ell(\mathcal{C}),$$

where the last inequality is given by Theorem 5.1. It completes the proof. □

Conjecture 5.4. [7, 24] *If* $\mathcal{C} \in MINT_2$ *then* $\chi_\ell(\mathcal{C}) = \chi(\mathcal{C})$.

Galvin [16] proved the conjecture for $\mathcal{C} = \mathcal{M} \cap \mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are partition matroids. More cases were studied in [7, 24, 19, 17].

6. Bounding $\Delta_\eta(\bigcap \mathcal{L})$ for partition matroids

In this section, we prove an upper bound on Δ_η, thereby also on χ_ℓ (by Theorem 5.1), of the intersection of k partition matroids. We conjecture that the result holds also for the intersection of general matroids, but in the general case we can only prove a weaker result — this will be done in the next section.

Theorem 6.1. *If* $\mathcal{L} = \{M_1, \dots, \mathcal{M}_k\} \in \mathcal{D}^k$ *is a set of* k *partition matroids, then*

$$\Delta_\eta(\bigcap \mathcal{L}) \leq k \max_{1\leq i\leq k} \Delta_r(\mathcal{M}_i). \tag{7}$$

Proof. Let $\mathcal{H} = \mathcal{K}(\mathcal{L})$, the k-partite hypergraph associated with $\mathcal{L}$ (see the definition above Theorem 1.5). Let $d = \max_{1\leq i\leq k} \Delta_r(\mathcal{M}_i)$. Then the maximum degree in $\mathcal{H}$ is d, hence putting weight $\frac{1}{d}$ on every edge of $\mathcal{H}$ yields a fractional matching, so $\nu^*(\mathcal{H}) \geq \frac{|E(\mathcal{H})|}{d}$. By Theorem 3.10 this implies $\eta(\mathcal{M}(\mathcal{H})) \geq \frac{|E(\mathcal{H})|}{kd}$, and thus $\frac{|E(\mathcal{H})|}{\eta(\mathcal{M}(\mathcal{H}))} \leq kd$. By the same token, this is true for any subhypergraph of $\mathcal{H}$, yielding (7). $\square$

The inequality in (7) is sharp, as shown by the following example.

Example 6.2. The k-uniform affine plane is obtained by removing a line L from the projective plane of uniformity $k+1$ (if such exists) and removing all vertices of L from the other lines. Thus the affine plane is a k-uniform hypergraph on k^2 vertices that is the union of $k+1$ matchings $M_1, \ldots, M_{k+1}$, each of size k, that are pairwise cross-intersecting, namely $e \cap f \neq \emptyset$ whenever e, f belong to different M_is. For example, the affine plane for $k = 2$ is K_4.

Choose one of the $k+1$ matchings, say M_1, and remove its edges (keeping the vertices). What remains is a k-partite hypergaph, which we name Q_k, whose sides are the edges of M_1. It is k-regular, namely each vertex is contained in k edges. Also, $|E(Q_k)| = k^2$. For example, $Q_2 = C_4$.

Let $\mathcal{L} = \mathcal{L}(Q_k)$. By the pairwise cross-intersecting property, we have $\eta(\bigcap \mathcal{L}) = \eta(\mathcal{M}(Q_k)) = 1$ and thus $\Delta_\eta(\bigcap \mathcal{L}) = |Q_k| = k^2$. On the other hand, the k-regularity of Q_k means $\Delta_r(\mathcal{M}_i) = k$ so that $\Delta_\eta(\bigcap \mathcal{L}) = k^2 = k \max_{1\leq i\leq k} \Delta_r(\mathcal{M}_i)$.

Combining Theorem 5.1, Theorem 6.1 and Theorem 4.4 yields the following corollary.

Corollary 6.3. *If $\mathcal{C}$ is the intersection of k partition matroids $\mathcal{M}_1, \ldots, \mathcal{M}_k$ on the same ground set, then $\chi_\ell(\mathcal{C}) \leq k \max_{1\leq i\leq k} \chi(\mathcal{M}_i)$.*

7. Bounding $\Delta_\eta(\bigcap \mathcal{L})$ for general matroids

7.1. **A bound.** As mentioned above, we suspect that Theorem 6.1 is also valid for k-tuples of general matroids, but we can only prove a weaker result.

Theorem 7.1. *Let $\mathcal{L} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\} \in \mathcal{D}^k$. Then*

$$\Delta_\eta(\bigcap \mathcal{L}) \leq (2k-1) \max_{1\leq i\leq k} \Delta_r(\mathcal{M}_i). \tag{8}$$

Combining with Theorem 5.1 and Theorem 4.4 this yields the following result.

Corollary 7.2. *Let $\mathcal{L} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\} \in \mathcal{D}^k$. Then*

$$\chi_\ell(\bigcap \mathcal{L}) \leq (2k-1) \max_{1\leq i\leq k} \chi(\mathcal{M}_i).$$

The proof of [2, Theorem 8.9] shows that for $\{\mathcal{M}_1, \mathcal{M}_2\} \in \mathcal{D}^2$, $\Delta_\eta(\mathcal{M}_1 \cap \mathcal{M}_2) \leq 2\max\big(\Delta(\mathcal{M}_1), \Delta(\mathcal{M}_2)\big)$, which together with Theorem 3.5 and Theorem 5.1 yields the following result.

Theorem 7.3. *For $\{\mathcal{M}_1, \mathcal{M}_2\} \in \mathcal{D}^2$, $\chi_\ell(\mathcal{M}_1 \cap \mathcal{M}_2) \leq 2\max_{1\leq i\leq 2} \chi(\mathcal{M}_i)$.*

In [8] this was strengthened to:

$$\chi_\ell(\mathcal{M}_1 \cap \mathcal{M}_2) \leq \chi(\mathcal{M}_1) + \chi(\mathcal{M}_2).$$

Conjecture 7.4. *For $\mathcal{L} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\} \in \mathcal{D}^k$ it holds*

$$\chi(\bigcap \mathcal{L}) \leq k \max_{1\leq i\leq k} \chi(\mathcal{M}_i).$$

In [2], it was conjectured that: $\chi(\mathcal{M}_1 \cap \mathcal{M}_2) \leq \max(\chi(\mathcal{M}_1), \chi(\mathcal{M}_2) + 1)$, and it is tempting to extend the conjecture to:

Conjecture 7.5. *For $\mathcal{L} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\} \in \mathcal{D}^k$ it holds*

$$\chi(\bigcap \mathcal{L}) \le (k-1) \max_{1 \le i \le k} \chi(\mathcal{M}_i) + 1.$$

We now go back to the proof of Theorem 7.1.

7.2. **A hypergraph Mayer-Vietoris inequality.** At the core of the proof of Theorem 7.1 stands a Mayer-Vietoris type inequality. To state it, we need the following operations on hypergraphs. For a hypergraph $\mathcal{H}$ and edge $e \in \mathcal{H}$, we write $\mathcal{H} - e$ for $\mathcal{H} \setminus \{e\}$. For $X \subseteq V$, the *contraction* hypergraph $\mathcal{H}/X$ has vertex set $V \setminus X$ and edge set $\{f \setminus X : f \in E, f \not\subseteq X\}$.

Theorem 7.6. *Let $\mathcal{H}$ be a hypergraph and let e be a containment-wise minimal edge of $\mathcal{H}$. Then*

$$\eta(\mathcal{I}(\mathcal{H})) \ge \min\Big(\eta(\mathcal{I}(\mathcal{H} - e)),\ \eta(\mathcal{I}(\mathcal{H}/e)) + |e| - 1\Big). \tag{9}$$

Proof. In Lemma 3.11 set $\mathcal{A} = \mathcal{I}(\mathcal{H})$ and $\mathcal{B} = 2^e * \mathcal{I}(\mathcal{H}/e)$. □

To state Theorem 7.7, we need the following definitions. Given a hypergraph $\mathcal{H}$, a sequence $\mathcal{K} = (e_1, \dots, e_p)$ of edges of $\mathcal{H}$ is *dominating* if for every vertex $v \in V \setminus \bigcup \mathcal{K}$ there exists an edge f of $\mathcal{H}$ such that $f \setminus \bigcup \mathcal{K} = \{v\}$. It is called *frugal* if $|e_i \setminus \bigcup_{\ell < i} e_\ell| > 1$ for every $1 \le i \le p$. Let $\gamma^E(\mathcal{H})$ be the minimum of $|\bigcup \mathcal{K}| - |\mathcal{K}|$ over all $\mathcal{K}$ that are both frugal and dominating (∞, if there is no such $\mathcal{K}$).

Theorem 7.7. *$\eta(\mathcal{I}(\mathcal{H})) \ge \gamma^E(\mathcal{H})$ for any hypergraph $\mathcal{H}$.*

Remark 7.8. This is a generalization of Theorem 3.12.

Proof of Theorem 7.7. Observe that if a hypergraph $\mathcal{G}$ has an edge of size 1, say $\{v\}$, then

$$\mathcal{I}(\mathcal{G}) = \mathcal{I}\Big(\mathcal{G}[V(\mathcal{G}) \setminus \{v\}]\Big). \tag{10}$$

To prove the theorem, we apply the following algorithm. Let $\mathcal{H}_0 = \mathcal{H}$. Suppose that the hypergraph $\mathcal{H}_j$ is defined for some $j \ge 0$. Delete all singleton edges and the vertices forming them. By (10) this does not change the independence complex. All edges of the resulting hypergraph $\mathcal{H}_j^*$ have sizes at least 2. If $E(\mathcal{H}_j^*) = \emptyset$ while $V(\mathcal{H}_j^*) \ne \emptyset$, we stop. In this case $\mathcal{I}(\mathcal{H}_j^*) = 2^{V(\mathcal{H}_j^*)}$ so $||\mathcal{I}(\mathcal{H}_j^*)|| \cong B^{|V(\mathcal{H}_j^*)|-1}$ and

$$\eta(\mathcal{I}(\mathcal{H}_j^*)) = \infty, \tag{11}$$

or we stop if $V(\mathcal{H}_j^*) = \emptyset$, in which case

$$\eta(\mathcal{I}(\mathcal{H}_j^*)) = 0. \tag{12}$$

If neither of these cases occurs, we proceed. Choose a containment-wise minimal edge f_j of $\mathcal{H}_j^*$. Then $|f_j| \ge 2$ and by Theorem 7.6

$$\eta(\mathcal{I}(\mathcal{H}_j^*)) \ge \min\Big(\eta(\mathcal{I}(\mathcal{H}_j^* - f_j)),\ \eta(\mathcal{I}(\mathcal{H}_j^*/f_j)) + |f_j| - 1\Big).$$

If $\eta(\mathcal{I}(\mathcal{H}_j^*)) \ge \eta(\mathcal{I}(\mathcal{H}_j^* - f_j))$, then we set $\mathcal{H}_{j+1} = \mathcal{H}_j^* - f_j$; otherwise we set $\mathcal{H}_{j+1} = \mathcal{H}_j^*/f_j$ and in this case, the edge f_j is marked as "contracted". Thus H_{j+1} is defined, and we go to the next step of the algorithm.

Clearly we will finally stop either because the edge set becomes empty and the vertex set is not empty, or the vertex set becomes empty. If the former happens, by (11), inductively we have

$$\eta(\mathcal{I}(\mathcal{H})) = \eta(\mathcal{I}(\mathcal{H}_0)) \geq \infty \geq \gamma^E(\mathcal{H})$$

so the theorem is true. If the latter happens, by (12), let e_i^* be those f_i that are contracted during the algorithm and assume they are $(e_1^*, \dots, e_p^*)$. Then by the construction,

$$\eta(\mathcal{I}(\mathcal{H})) = \eta(\mathcal{I}(\mathcal{H}_0)) \geq \sum_{i=1}^{p}(|e_i^*| - 1). \tag{13}$$

Furthermore, since the deletions of vertices or edges does not change the remaining edges, while contracting f_i is deleting f_i from other edges, then we know that there exists edges $e_1, \dots, e_p$ of $\mathcal{H}_0 = \mathcal{H}$ such that for each $1 \leq i \leq p$

$$e_i \setminus \cup_{\ell<i} e_\ell = e_i^*.$$

Let $\mathcal{K} = (e_1, \dots, e_p)$. Since $|e_i^*| \geq 2$, we have that $\mathcal{K}$ is frugal. And for every $v \in V(\mathcal{H}) \setminus \bigcup \mathcal{K}$, v is a deleted vertex during the algorithm, which means during the algorithm there exists an edge of size one consisting of v. Thus for the same reason as above, there exists an edge f of $\mathcal{H}$ such that $f \setminus \bigcup \mathcal{K} = \{v\}$. Therefore $\mathcal{K}$ is dominating. And we have

$$|\bigcup \mathcal{K}| - |\mathcal{K}| = \sum_{i=1}^{p} |e_i \setminus \cup_{\ell<i} e_\ell| - p = \sum_{i=1}^{p}(|e_i^*| - 1).$$

Therefore by (13), $\eta(\mathcal{I}(\mathcal{H})) \geq \gamma^E(\mathcal{H})$, as desired. □

7.3. **Proof of Theorem 7.1.**

Observation 7.9 (e.g., Lemma 1.4.8 in [29])**.** *For any circuit C in a matroid $\mathcal{M}$, any $v \in C$, and any subset T of the ground set of $\mathcal{M}$,*

$$span_{\mathcal{M}}(T \cup C) = span_{\mathcal{M}}(T \cup C \setminus \{v\}).$$

The *circuit hypergraph $CIRC(\mathcal{M})$* of a matroid $\mathcal{M}$ is the set of circuits in $\mathcal{M}$. By the definition of "circuit", if $\mathcal{H} = CIRC(\mathcal{M})$ then $\mathcal{M} = \mathcal{I}(\mathcal{H})$.

Proof of Theorem 7.1. Let $t = \max_{1\leq i\leq k} \Delta_r(\mathcal{M}_i)$, $\mathcal{C} = \cap_{i=1}^k \mathcal{M}_i$, and V be the common ground set of the matroids. First we prove that

$$\frac{|V|}{\eta(\mathcal{C})} \leq (2k-1)t.$$

Let $\mathcal{H}_i = CIRC(\mathcal{M}_i)$ for $1 \leq i \leq k$ and $\mathcal{H} = \cup_{i=1}^k \mathcal{H}_i$. Then

$$\mathcal{C} = \mathcal{I}(\mathcal{H}).$$

We may assume that $\gamma^E(\mathcal{H})$ is finite, otherwise by Theorem 7.7, $\eta(\mathcal{C}) = \infty$ and we are done. Let $\mathcal{K} = (e_1, \dots, e_p)$ be a dominating and frugal sequence of edges of $\mathcal{H}$ for which

$$\gamma^E(\mathcal{H}) = |\bigcup \mathcal{K}| - |\mathcal{K}| = \sum_{j=1}^{p} |e_j \setminus \cup_{\ell<j} e_\ell| - p = \sum_{j=1}^{p}(|e_j \setminus \cup_{\ell<j} e_\ell| - 1). \tag{14}$$

Since $\mathcal{K}$ is dominating, for any $v \in V \setminus \bigcup \mathcal{K}$ there exists an edge $f \in \mathcal{H}$ such that

$$f \setminus \bigcup \mathcal{K} = \{v\}.$$

Let i be such that $f \in \mathcal{H}_i = CIRC(\mathcal{M}_i)$. Then $v \in span_{\mathcal{M}_i}(f \setminus \{v\})$, and then

$$v \in \bigcup \mathcal{K} \cup f \subseteq span_{\mathcal{M}_i}(\bigcup \mathcal{K} \cup f \setminus \{v\}) = span_{\mathcal{M}_i}(\bigcup \mathcal{K}).$$

Therefore

$$V \subseteq \cup_{i=1}^{k} span_{\mathcal{M}_i}(\bigcup \mathcal{K}). \tag{15}$$

For each $1 \leq j \leq p$, noting that e_j is an edge of $\mathcal{H} = \cup_{i=1}^{k} \mathcal{H}_i$, let

$$L_j = \{i \mid e_j \text{ is an edge of } \mathcal{H}_i\}.$$

For each $1 \leq i \leq k$ and each index j such that $i \in L_j$, we choose an element $v_{i,j} \in e_j \setminus \cup_{\ell<j} e_\ell$. Recall that $i \in L_j$ means that e_j is a circuit in $\mathcal{M}_i$. Hence, by Observation 7.9, we can delete $(v_{i,j})_{\{j:i\in L_j\}}$ one by one, in descending order of j, to obtain:

$$span_{\mathcal{M}_i}\Big(\big(\cup_{j:i\in L_j} e_j\big) \cup \big(\cup_{j:i\notin L_j} e_j\big)\Big) = span_{\mathcal{M}_i}\Big(\big(\cup_{j:i\in L_j} (e_j \setminus \{v_{i,j}\})\big) \cup \big(\cup_{j:i\notin L_j} e_j\big)\Big)$$

and then

$$\begin{aligned}
&span_{\mathcal{M}_i}(\bigcup \mathcal{K}) \\
=&span_{\mathcal{M}_i}\Big(\big(\cup_{j:i\in L_j} e_j\big) \cup \big(\cup_{j:i\notin L_j} e_j\big)\Big) \\
=&span_{\mathcal{M}_i}\Big(\big(\cup_{j:i\in L_j} (e_j \setminus \{v_{i,j}\})\big) \cup \big(\cup_{j:i\notin L_j} e_j\big)\Big) \\
=&span_{\mathcal{M}_i}\bigg(\Big(\cup_{j:i\in L_j} \big(e_j \setminus (\cup_{\ell<j} e_\ell \cup \{v_{i,j}\})\big)\Big) \cup \Big(\cup_{j:i\notin L_j} (e_j \setminus \cup_{\ell<j} e_\ell)\Big)\bigg),
\end{aligned}$$

which has size at most

$$t\Big(\sum_{j:i\in L_j} (|e_j \setminus \cup_{\ell<j} e_\ell| - 1) + \sum_{j:i\notin L_j} |e_j \setminus \cup_{\ell<j} e_\ell|\Big). \tag{16}$$

Let $f_j = e_j \setminus \cup_{\ell<j} e_\ell$. By the frugality, $|f_j| = |e_j \setminus \cup_{\ell<j} e_\ell| \geq 2$, so that

$$|f_j| \leq 2(|f_j| - 1). \tag{17}$$

Since for each j, $e_j \in \mathcal{H}_i$ for some $i \in L_j$, there are at most $k-1$ many indices i' in $[k]$ such that $i' \notin L_j$ Then by (15), (16), and (17) we have

$$\begin{aligned}
|V| &\leq t\sum_{i=1}^{k}\Big(\sum_{j:i\in L_j}(|f_j|-1)+\sum_{j:i\notin L_j}|f_j|\Big)\\
&\leq t\sum_{i=1}^{k}\Big(\sum_{j:i\in L_j}(|f_j|-1)+\sum_{j:i\notin L_j}2(|f_j|-1)\Big)\\
&= t\sum_{j=1}^{p}\Big(\sum_{i:i\in L_j}(|f_j|-1)+\sum_{i:i\notin L_j}2(|f_j|-1)\Big)\\
&\leq t\sum_{j=1}^{p}\Big((|f_j|-1)+(k-1)2(|f_j|-1)\Big)\\
&= (2k-1)t\sum_{j=1}^{p}(|f_j|-1).
\end{aligned}$$

On the other hand, by (14)

$$\eta(\mathcal{C})=\eta(\mathcal{I}(\mathcal{H}))\geq\gamma^E(\mathcal{H})=\sum_{j=1}^{p}(|e_j\setminus\cup_{\ell<j}e_\ell|-1)=\sum_{j=1}^{p}(|f_j|-1).$$

We have thus shown

$$\frac{|V(\mathcal{C})|}{\eta(\mathcal{C})}\leq(2k-1)t.$$

Applying this argument to $\mathcal{H}[S]$, for any $S\subseteq V$, yields

$$\Delta_\eta(\mathcal{C})=\max_{\emptyset\neq S\subseteq V}\frac{|S|}{\eta(\mathcal{C}[S])}\leq(2k-1)t,$$

proving the theorem. □

8. Fractional coloring and polytopes

We turn to ($\vec{w}$-)fractional coloring (see Definition 1.2). In this section and in its two successors we link ($\vec{w}$-)fractional colorings to polytopes. The aim is to generalize the following result, proved in [2], to more than two matroids, and to vertex-weighted matroids.

Theorem 8.1. *For any pair of matroids $\mathcal{M}$ and $\mathcal{N}$ on the same ground set,*

$$\chi^*(\mathcal{M}\cap\mathcal{N})=\max(\chi^*(\mathcal{M}),\ \chi^*(\mathcal{N})). \tag{18}$$

In fact, this was merely testimony to ignorance — the authors were unaware of Theorem 2.1, of which it is a direct corollary.

Observation 8.2. *For any complex $\mathcal{C}$, $\chi^*(\mathcal{C})\leq\frac{1}{t}$ if and only if $t\vec{1}\in P(\mathcal{C})$.*

Proof. $t\vec{1}\in P(\mathcal{C})$ means that $t\vec{1}\leq\sum_{S\in\mathcal{C}}\lambda_S\mathbf{1}_S$ where $\lambda_S\geq 0$ and $\sum_{S\in\mathcal{C}}\lambda_S=1$. It is equivalent to $\vec{1}\leq\sum(\frac{\lambda_S}{t})\mathbf{1}_S$, meaning that $\chi^*(\mathcal{C})\leq\sum\frac{\lambda_S}{t}=\frac{1}{t}$. □

Combined with Theorem 2.1, this yields Theorem 8.1. There is nothing special about the vector $\vec{1}$ — the same argument works for every vector $\vec{w} \in \mathbb{R}^V$. Below, $\vec{w}$ always denotes such a vector. It is also viewed as a weight function on V, whence the arrow above is sometimes omitted.

For a closed down polytope $Z \subseteq \mathbb{R}^V_{\geq 0}$, let

$$\psi(Z, \vec{w}) := \min(\{t \in \mathbb{R}^+ \mid \vec{w}/t \in Z\}).$$

Theorem 8.3. $\chi^*(\mathcal{C}, \vec{w}) = \psi(P(\mathcal{C}), \vec{w})$.

Proof. Let $\psi(P(\mathcal{C}), \vec{w}) = t$. Then $\frac{\vec{w}}{t} \in P(\mathcal{C})$ implies that there exists $\lambda_S \geq 0$ for all $S \in \mathcal{C}$ so that $\vec{w} = \sum_{S \in \mathcal{C}} t\lambda_S \mathbf{1}_S$, where $\sum_{S \in \mathcal{C}} \lambda_S = 1$. Then, the function $f : \mathcal{C} \to \mathbb{R}_{\geq 0}$ defined as $f(S) := t\lambda_S \geq 0$ satisfies that for each $v \in V(\mathcal{C})$, $\sum_{S \in \mathcal{C}: v \in S} f(S) = \sum_{S \in \mathcal{C}} t\lambda_S \mathbf{1}_S(v) = w(v)$. Thus f is an $\vec{w}$-fractional coloring. Since $\sum_{S \in \mathcal{C}} f(S) = \sum_{S \in \mathcal{C}} t\lambda_S = t$, we have $\chi^*(\mathcal{C}, \vec{w}) \leq t = \psi(P(\mathcal{C}), \vec{w})$.

For the other direction, let $f : \mathcal{C} \to \mathbb{R}_{\geq 0}$ be an $\vec{w}$-fractional coloring satisfying $\sum_{S \in \mathcal{C}} f(S) = \chi^*(\mathcal{C}, \vec{w}) = a$. Let $\lambda_S = \frac{1}{a} f(S)$ for every $S \in \mathcal{C}$, and consider the function $g = \sum_{S \in \mathcal{C}} \lambda_S \mathbf{1}_S : V(\mathcal{C}) \to \mathbb{R}_{\geq 0}$. This is a convex combination of the $(\mathbf{1}_S)_{S \in \mathcal{C}}$ and thus $\vec{g} \in P(\mathcal{C})$. Since f is an $\vec{w}$-fractional coloring, $g(v) = \sum_{S \in \mathcal{C}: v \in S} \lambda_S = \sum_{S \in \mathcal{C}: v \in S} \frac{1}{a} f(S) \geq \frac{w(v)}{a}$ for each $v \in V(\mathcal{C})$. So, $0 \leq \frac{\vec{w}}{a} \leq \vec{g}$. Since $P(\mathcal{C})$ is closed-down, we have $\frac{\vec{w}}{a} \in P(\mathcal{C})$ and thus, $\psi(P(\mathcal{C}), \vec{w}) \leq a = \chi^*(\mathcal{C}, \vec{w})$. □

The theorem says that $\chi^*(\mathcal{C}, \vec{w})$ is the smallest t for which $\vec{w}/t \in P(\mathcal{C})$. On the other hand, for $\mathcal{L} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\} \in \mathcal{D}^k$, to enter $\cap_{i=1}^k P(\mathcal{M}_i)$ along the direction $\vec{w}$ towards the origin, you need to enter all $P(\mathcal{M}_i)$s. This is summarized in the following corollary, where as usual, $\lambda A = \{\lambda \vec{a} \mid \vec{a} \in A\}$.

Corollary 8.4. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be complexes on V and $\mathcal{C} = \cap_{i=1}^k \mathcal{C}_i$. The following are equivalent for any $\lambda > 0$:*

(1) $\lambda P(\mathcal{C}) \supseteq \cap_{i=1}^k P(\mathcal{C}_i)$.

(2) $\chi^*(\mathcal{C}, \vec{w}) \leq \lambda \max_{1 \leq i \leq k} \chi^*(\mathcal{C}_i, \vec{w})$ *for every* $\vec{w} \in \mathbb{R}^V_{\geq 0}$.

In the spirit of Corollary 7.2 and Conjecture 7.4, we can ask for bounds on the ratio between the fractional chromatic number of the intersection of the matroids and the fractional chromatic number of each matroid individually. It is a viewpoint that gives rise to Conjecture 2.3. Another motivation of Conjecture 2.3 is discussed in detail in Section 10. Conjecture 2.3 is known for general k if $k-1$ is replaced by k (see Corollary 10.15), and for $k = 2$ as it is (this follows from Corollary 8.4 and Theorem 2.1). In the next section we apply a result from [26] to show it for $k = 3$.

9. Two more polytopes and polytope ratios

This section discusses an equivalent version of Conjecture 2.3, involving polytopes.

Given a complex $\mathcal{C}$, let

$$Q(\mathcal{C}) = \{f \in \mathbb{R}^{V(\mathcal{C})}_{\geq 0} \mid f[S] \leq rank_{\mathcal{C}}(S) \text{ for every } S \subseteq V(\mathcal{C})\}.$$

Theorem 9.1. [11] *For a matroid $\mathcal{M}$ we have $P(\mathcal{M}) = Q(\mathcal{M})$.*

For general complexes the two polytopes can be arbitrarily far apart. For functions $f, g : V \to \mathbb{R}$, $f \cdot g$ is the inner product $\sum_{v \in V} f(v)g(v)$.

Observation 9.2. *For any $\lambda > 0$, there exists a complex $\mathcal{C}$ such that $\lambda P(\mathcal{C}) \not\supseteq Q(\mathcal{C})$.*

Proof. We construct a complex $\mathcal{C}$ and a vector $v \in \mathbb{R}^V_{\geq 0}$ such that $v \in Q(\mathcal{C}) \setminus \lambda P(\mathcal{C})$.

Let $k \in \mathbb{Z}^+$ be such that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > \lambda$, and let $n = 2 + 3 + \cdots + k$. We construct the complex $\mathcal{C}$ on $V = [n]$ in the following way. Let

$$v = (\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, ..., \frac{1}{k}, ..., \frac{1}{k}) \in \mathbb{R}^n,$$

where $\frac{1}{i}$ occures i times for each $2 \leq i \leq k$. Let

$$\mathcal{C} = \{A \subseteq [n] : v \cdot \mathbf{1}_A \leq 1\}.$$

Since any vector $u \in P(\mathcal{C})$ is a convex combination of $\{\mathbf{1}_A \mid A \in \mathcal{C}\}$, then $v \cdot u \leq 1$. Therefore $u \in \lambda P(\mathcal{C})$ implies $v \cdot u \leq \lambda$. Since

$$v \cdot v = \sum_{i=2}^{k} \frac{1}{i} \cdot \frac{1}{i} \cdot i > \lambda,$$

we have $v \notin \lambda P(\mathcal{C})$.

On the other hand, for $S \subseteq [n]$, let r be the smallest integer such that $-1 < v[S] \leq r$. Then S must have at least r members with v-value at most $\frac{1}{r}$: if not,

$$v[S] = v[\{j \in S : v(j) > 1/r\}] + v[\{j \in S : v(j) \leq 1/r\}] \leq \sum_{i=2}^{r-1} \frac{1}{i} \cdot i + r\frac{1}{r} \leq r - 1,$$

a contradiction. Therefore the r members form a face of $\mathcal{C}$ and we have $rank_{\mathcal{C}}(S) \geq r \geq v[S]$, which implies $v \in Q(\mathcal{C})$. □

$P(\mathcal{C}) \neq Q(\mathcal{C})$ is possible even in flag complexes, as the following example shows.

Example 9.3. Define a complex $\mathcal{C}$ on $V = \{x_1, x_2, \ldots, x_9, y_1, y_2, y_3, z_1, z_2, z_3\}$ and let the maximal faces of $\mathcal{C}$ be $\{y_1, y_2, y_3\}$, $\{z_1, z_2, z_3\}$, and $\{y_i, x_{3(i-1)+j}\}$, $\{z_j, x_{3(i-1)+j}\}$ for $1 \leq i, j \leq 3$. It can be verified that this complex is 2-determined so it is a flag complex. Therefore by Theorem 1.5, $\mathcal{C}$ is the intersection of partition matroids.

Consider the vector w such that $w(x_i) = \frac{1}{9}$ for each $1 \leq i \leq 9$ and $w(y_j) = w(z_j) = \frac{1}{4}$ for each $1 \leq j \leq 9$. Then we claim that

$$w \in Q(\mathcal{C}) \setminus P(\mathcal{C}).$$

To see that $w \in Q(\mathcal{C})$, for every $U \subseteq V$ such that $rank_{\mathcal{C}}(U) = 1$, the maximal such U in containment relation must be $\{x_t\}_{1 \leq t \leq 9}$ or consist of four elements of $\{x_t\}_{1 \leq t \leq 9}$, one elements of $\{y_1, y_2, y_3\}$, and one element of $\{z_1, z_2, z_3\}$. In either case we have $w[U] \leq 1$. Since $rank(\mathcal{C}) = 3$, for every set U with $rank_{\mathcal{C}}(U) = 3$, we have $w[U] \leq w[V] \leq \frac{1}{9} \cdot 9 + \frac{1}{4} \cdot 6 \leq 3$. The maximal rank 2 set in $\mathcal{C}$ has the form

$$\{y_i, y_{i'}, z_j, z_{j'}, x_1, x_2, \ldots, x_9\}$$

for $i \neq i'$ and $j \neq j'$, which has w-weight $\frac{1}{9} \cdot 9 + \frac{1}{4} \cdot 4 \leq 2$, which implies for any U with $rank_{\mathcal{C}}(U) = 2$, $w[U] \leq 2$. Therefore $w \in Q(\mathcal{C})$.

To see that $w \notin P(\mathcal{C})$, we are going to prove that w cannot be written as a convex combination $\sum \lambda_i \mathbf{1}_{S_i}$ for $S_i \in \mathcal{C}$. First, the coefficients of $\{y_1, y_2, y_3\}$ and $\{z_1, z_2, z_3\}$ cannot be positive, since the coefficients of the faces containing any x_i for $1 \leq i \leq 9$ should sum up to $\frac{1}{9} \cdot 9 = 1$. Second, for the sum of the coefficients of the faces of the form $\{x_p, y_i\}$ or $\{x_q, z_j\}$, when considering from the $\{x_t\}_{1 \leq t \leq 9}$ side, the sum is at most $\frac{1}{9} \cdot 9 = 1$, but it should be at least $\frac{1}{4} \cdot 6 = \frac{3}{2}$ when considering from the $\{y_1, y_2, y_3, z_1, z_2, z_3\}$ side, which is impossible. Therefore $w \notin P(\mathcal{C})$.

There is a third polytope associated with k-tuples of matroids. For $\mathcal{L} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\} \in \mathcal{D}^k$, let

$$R(\mathcal{L}) = \cap_{i=1}^{k} P(\mathcal{M}_i).$$

Theorem 9.1 implies

$$P(\bigcap \mathcal{L}) \subseteq Q(\bigcap \mathcal{L}) \subseteq R(\mathcal{L}). \tag{19}$$

Theorem 2.1 says that for $k = 2$ equality holds throughout. By Observation 9.2 and Example 9.3 (see also Example 10.26 below) this is no longer true for $k > 2$.

Definition 9.4. Given closed-down polytopes A, B in $\mathbb{R}^V_{\geq 0}$, let

$$B : A = \min\{t \in \mathbb{R}_{\geq 0} \mid tA \supseteq B\}$$

(here, as usual, $\min \emptyset = \infty$).

By Corollary 8.4, another formulation of Conjecture 2.3 is the following.

Conjecture 9.5. *For any $\mathcal{L} \in \mathcal{D}^k$,*

$$R(\mathcal{L}) : P(\bigcap \mathcal{L}) \leq k - 1.$$

10. Polytope ratios and weighted matroids

To study Conjecture 9.5, we need vertex-weighted versions of matchings and cooperative coverings. For an algorithmic viewpoint of this subject see [13]. Throughout this section we have at hand a weight function $w \in \mathbb{R}^V_{\geq 0}$.

Given a matroid $\mathcal{M}$ let $\bar{\nu}_w(\mathcal{M})$ be the maximum sum of weights in an edge of $\mathcal{M}$. For a function $f : V \to \mathbb{R}$ the *$\mathcal{M}$-span* $f^{\mathcal{M}}$ of f is a function, defined by

$$f^{\mathcal{M}}(v) = \max_{T \subseteq V : v \in span_{\mathcal{M}}(T)} \min_{u \in T} f(u).$$

Otherwise put,

$$f^{\mathcal{M}}(v) \geq t \Leftrightarrow v \in span_{\mathcal{M}}\{u \mid f(u) \geq t\}.$$

In particular, for $A \subseteq V$ we have $\mathbf{1}_A^{\mathcal{M}} = \mathbf{1}_{span_{\mathcal{M}}(A)}$. A function f is said to be *w-spanning* if $f^{\mathcal{M}} \geq w$. Let $\bar{\tau}^*_w(\mathcal{M}) = \min\{f[V] : f^{\mathcal{M}} \geq w\}$. A folklore result (see for example [2]) is $\bar{\tau}^*_w(\mathcal{M}) = \bar{\nu}_w(\mathcal{M})$.

Lemma 10.1. *Let $\mathcal{M}$ be a matroid, $\vec{x} \in P(\mathcal{M})$ and $f \in \mathbb{R}^V_{\geq 0}$. Then $f^{\mathcal{M}} \cdot \vec{x} \leq f[V]$.*

Proof. By the definition of $P(\mathcal{M})$ it suffices to prove the case that $x = \mathbf{1}_S$ for $S \in \mathcal{M}$. Assume $Im(f) = \{a_1, \ldots, a_p\}$ for $0 \leq a_1 < \cdots < a_p$. For $\ell \in [p]$, let $Z_\ell = f^{-1}(a_\ell)$ and $U_\ell = span_{\mathcal{M}}(\cup_{\ell \leq t \leq p} Z_t)$. We prove the theorem by induction on p.

When $p = 1$, $f^{\mathcal{M}} \cdot x = a_1|S \cap U_1|$. Since $S \in \mathcal{M}$ and $U_1 = span_{\mathcal{M}}(Z_1)$,

$$|S \cap U_1| \leq rank_{\mathcal{M}}(S \cap U_1) \leq |Z_1|,$$

which implies that $f^{\mathcal{M}} \cdot x \leq a_1|Z_1| = f[V]$.

For $p \geq 2$, define a function f_1 such that for $v \in \cup_{\ell=1}^{p-1} Z_\ell$, $f_1(v) = f(v)$, and for $v \in Z_p$, $f_1(v) = a_{p-1}$. Then

$$f^{\mathcal{M}} \cdot x = f_1^{\mathcal{M}} \cdot x + (a_p - a_{p-1})|S \cap U_p|.$$

By induction hypothesis, $f_1^{\mathcal{M}} \cdot x \leq f_1[V]$. On the other hand, $(a_p - a_{p-1})|S \cap U_p| \leq (a_p - a_{p-1}) rank_{\mathcal{M}}(U_p) \leq (a_p - a_{p-1})|Z_p|$. Combining these implies that $f^{\mathcal{M}} \cdot x \leq f_1[V] + (a_p - a_{p-1})|Z_p| = f[V]$. □

The concept of w-spanning has a cooperative version in $\mathcal{D}^k$. Given $\mathcal{L} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\} \in \mathcal{D}^k$, a k-tuple $(f_1, f_2, \dots, f_k)$ of functions in $\mathbb{R}^V_{\geq 0}$ is a *matroidal fractional $\vec{w}$-cooperative cover* if $\sum_{i=1}^k f_i^{\mathcal{M}_i}(v) \geq w(v)$ for every $v \in V$. The *matroidal fractional $\vec{w}$-covering number* $\bar{\tau}^*_w(\mathcal{L})$ is the minimum of $\sum_{i=1}^k f_i[V]$ over all matroidal fractional $\vec{w}$-cooperative covers $(f_1, f_2, \dots, f_k)$. The *matroidal cooperative covering $\vec{w}$-number* $\bar{\tau}_w(\mathcal{L})$ is the minimum of $\sum_{i=1}^k f_i[V]$ over all fractional $\vec{w}$-cooperative covers $(f_1, f_2, \dots f_k)$, where each f_i has integral values. When $\vec{w} = \vec{1}$, we omit the subscript $\vec{1}$.

Remark 10.2. The notational rule here and later is that a parameter z^*_w stands for the fractional w-weighted version, and z_w stands for the integral w-weighted version. We omit the mention of w when $\vec{w} = \vec{1}$.

Remark 10.3. This deviates from the notation in [2], where our $\bar{\tau}^*_w$ is denoted by τ_w. Please also note the difference between this and fractional weighted covering and matching numbers in hypergraphs (see Definition 10.17), where the weights are on edges, rather than on vertices.

The *matroidal fractional w-matching number* $\bar{\nu}^*_w(\mathcal{L})$ is the maximum of $\vec{x} \cdot \vec{w}$ over all $\vec{x} \in R(\mathcal{L})$. $\bar{\nu}_w(\mathcal{L})$, $\bar{\nu}^*(\mathcal{L})$, and $\bar{\nu}(\mathcal{L})$ are similarly defined.

When $w \equiv 1$ and the functions are integral we have $\bar{\nu}^* = \bar{\nu}$ and $\bar{\tau}^* = \bar{\tau}$, as defined in the introduction.

Theorem 10.4. *If $\mathcal{L} \in \mathcal{D}^k$ for some k, then $\bar{\tau}^*_w(\mathcal{L}) = \bar{\nu}^*_w(\mathcal{L})$.*

Proof. The inequality $\bar{\tau}^*_w(\mathcal{L}) \geq \bar{\nu}^*_w(\mathcal{L})$ is a straightforward corollary of Lemma 10.1: Indeed, let $\vec{x} \in R(\mathcal{L}) = \cap_{i=1}^k P(\mathcal{M}_i)$ and let $(f_1, \dots, f_k)$ be a matroidal fractional w-cooperative cover (so that $\sum_{i=1}^k f_i^{\mathcal{M}_i} \geq w$) satisfying $\sum_{i=1}^k f_i[V] = \bar{\tau}^*_w(\mathcal{L})$. Then by Lemma 10.1,

$$w \cdot x \leq \sum_{i=1}^k f^{\mathcal{M}_i} \cdot x \leq \sum_{i=1}^k f_i[V] = \bar{\tau}^*_w(\mathcal{L}).$$

As this is true for all $x \in R(\mathcal{L})$, it follows that $\bar{\nu}^*_w(\mathcal{L}) \leq \bar{\tau}^*_w(\mathcal{L})$.

For the proof of the inverse inequality, note that by Theorem 9.1 the dual program of $\bar{\nu}^*_w(\mathcal{L})$ is

$$\bar{\nu}^*_w(\mathcal{L}) = \min \sum_{1 \leq i \leq k, U \subseteq V} rank_{\mathcal{M}_i}(U) y_i(U)$$

subject to

$$\sum_{1 \leq i \leq k, U \subseteq V : v \in U} y_i(U) \geq w(v) \text{ for every } v \in V,$$
$$y_i(U) \geq 0 \text{ for every } U \subseteq V \text{ and } 1 \leq i \leq k.$$

Let $(y_1, \dots, y_k)$ be an optimal solution of the dual LP, minimizing

$$\sum_{1 \leq i \leq k, U \subseteq V} y_i(U) \cdot |U||V \setminus U|. \tag{20}$$

Let $\mathcal{S}_i \subseteq 2^V$ be the support of y_i for $1 \le i \le k$. We claim that for every $i \in [k]$, the collection $\mathcal{S}_i$ is a chain, i.e., for two distinct $S, T \in \mathcal{S}_i$, either $S \subseteq T$ or $T \subseteq S$. Suppose not, i.e., there exist $S, T \in \mathcal{S}_i$ such that S and T are incomparable in inclusion relation. Let $\alpha := \min\Big(y_i(S),\ y_i(T)\Big)$ and define

$$y_i^*(U) = \begin{cases} y_i(U) - \alpha & \text{if } U \in \{S, T\}, \\ y_i(U) + \alpha & \text{if } U \in \{S \cap T, S \cup T\}, \\ y_i(U) & \text{otherwise.} \end{cases}$$

Since

$$\mathbf{1}_S + \mathbf{1}_T = \mathbf{1}_{S\cap T} + \mathbf{1}_{S \cup T},$$

$(y_1, \ldots, y_{i-1}, y_i^*, y_{i+1}, \ldots, y_k)$ remains a feasible solution of the dual LP. And since

$$rank_{\mathcal{M}_i}(S) + rank_{\mathcal{M}_i}(T) \ge rank_{\mathcal{M}_i}(S \cap T) + rank_{\mathcal{M}_i}(S \cup T),$$

it remains optimal. However it is easy to check that for the incomparable S and T (see, e.g., [31, Theorem 2.1])

$$|S||S^c| + |T||T^c| > |S \cap T||(S \cap T)^c| + |S \cup T||(S \cup T)^c|,$$

where $U^c := V \setminus U$, therefore the new optimal solution has a smaller sum (20), a contradiction. Therefore $\mathcal{S}_i$ is a chain.

Assume $S_{i,1} \subseteq \cdots \subseteq S_{i,\ell_i}$ is the chain in $\mathcal{S}_i$. By the independence augmentation property of $\mathcal{M}_i$, there exists a chain of independent sets $I_{i,1} \subseteq \cdots \subseteq I_{i,\ell_i}$ of $\mathcal{M}_i$ such that $I_{i,t} \subseteq S_{i,t}$ and $rank_{\mathcal{M}_i}(S_{i,t}) = |I_{i,t}|$ for each $t \in [\ell_i]$. We set

$$f_i(v) := \sum_{t=1}^{\ell_i} 1_{\{v \in I_{i,t}\}} y_i(S_{i,t}),$$

where $1_{\{v \in I_{i,t}\}}$ is the characteristic function of the event $\{v \in I_{i,t}\}$. Then

$$f_i[V] = \sum_{t=1}^{\ell_i} rank_{\mathcal{M}_i}(S_{i,t}) y_i(S_{i,t}) = \sum_{U \subseteq V} rank_{\mathcal{M}_i}(U) y_i(U),$$

therefore

$$\sum_{i=1}^{k} f_i[V] = \bar{\nu}_w^*.$$

It remains to prove that $(f_1, \ldots, f_k)$ is a matroidal fractional $\vec{w}$-cooperative cover. Indeed, for any $v \in V$ and $i \in [k]$, in the chain $S_{i,1} \subseteq \cdots \subseteq S_{i,\ell_i}$ of $\mathcal{S}_i$, assume $\ell \in [\ell_i]$ is the minimum index such that $v \in S_{i,\ell}$. Then

$$\sum_{U \subseteq V : v \in U} y_i(U) = \sum_{\ell \le t \le \ell_i} y_i(S_{i,t}). \tag{21}$$

On the other hand, $v \in S_{i,\ell} \subseteq span_{\mathcal{M}_i}(I_{i,\ell})$ and $\min_{u \in I_{i,\ell}} f(u) \ge \sum_{\ell \le t \le \ell_i} y_i(S_{i,t})$ by the choice of the chain $(I_{i,t})_t$ and the definition of f_i. Therefore

$$f^{\mathcal{M}_i}(v) \ge \sum_{\ell \le t \le \ell_i} y_i(S_{i,t}),$$

which together with (21) implies

$$\sum_{i=1}^{k} f^{\mathcal{M}_i}(v) \ge \sum_{1 \le i \le k, U \subseteq V, v \in U} y_i(U) \ge w(v).$$

This completes the proof that $\bar\tau^*_w \le \bar\nu^*_w$, and thereby the theorem. $\square$

Corollary 10.5. $\bar\nu_w(\mathcal{L}) \le \bar\nu^*_w(\mathcal{L}) = \bar\tau^*_w(\mathcal{L}) \le \bar\tau_w(\mathcal{L})$.

The gaps between these parameters are closely related to the ratios between the polytopes, as reflected in the following two results.

Theorem 10.6. $R(\mathcal{L}) : P(\bigcap \mathcal{L}) = \max_{w\in\mathbb{R}^V_{\ge 0}} \bar\tau^*_w(\mathcal{L}) : \bar\nu_w(\mathcal{L})$.

Proof. Let $R = R(\mathcal{L})$ and $P = P(\bigcap \mathcal{L})$. Let $\lambda = \max_{w\in\mathbb{R}^V_{\ge 0}} \bar\tau^*_w(\mathcal{L}) : \bar\nu_w(\mathcal{L})$ and $\gamma = R(\mathcal{L}) : P(\bigcap \mathcal{L})$.

Let us first show that $\gamma \le \lambda$. Assume for contradiction that $\gamma > \lambda$. This means that there exists a vector $\vec z \in R(\mathcal{L}) \setminus \lambda P(\bigcap \mathcal{L})$. We may clearly assume that each coordinate of $\vec z$ is strictly positive: suppose not, say $\vec z(j) = 0$. Since R is convex and $\{j\} \in \bigcap \mathcal{L}$, then $\alpha\vec z + (1-\alpha)\mathbf{1}_{\{j\}} \in R$ for every $\alpha \in [0,1]$. Since $\mathbf{1}_{\{j\}} \in \lambda P$ (as $\lambda \ge 1$) and λP is closed, then there exists $\alpha_0 \in (0,1)$ such that $\alpha_0\vec z + (1-\alpha_0)\mathbf{1}_{\{j\}} \in R\setminus\lambda P$.

On the other hand, the convex polytope λP is the intersection of a collection of half-spaces $\{\vec x \mid \vec w_i \cdot \vec x \le a_i\}$ for $i \in [m]$, which means that the polytope P is the intersection of the collection

$$\{\vec y \mid \lambda\vec w_i \cdot \vec y \le a_i\} \tag{22}$$

for $i \in [m]$. Thus there exists some i such that $\vec w_i \cdot \vec z > a_i$. Since $\{j\} \in \bigcap \mathcal{L}$ for every $j \in V$, λP contains a standard simplex, so there exists some $t \in (0,1)$ such that $t\vec z \in \lambda P$ and $\vec w_i \cdot t\vec z = a_i$. Then $\vec w_i \ge 0$: otherwise suppose $\vec w_i(j) < 0$ for some j, then $\vec w_i \cdot (t\vec z - se_j) > a_i$ for every $0 < s < (t\vec z)(j)$, contradicting the fact that λP is closed down.

For $\mathcal{L} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\}$, an integral vector in $R(\mathcal{L}) = \cap_{i=1}^k Q(\mathcal{M}_i)$ is the characteristic vector of a face in $\mathcal{C}$, and vice versa. Recalling that $P(\mathcal{C})$ is the convex hull of the characteristic vectors of faces in $\mathcal{C}$ and using (22) for the last inequality in the following, the above argument implies that

$$\bar\tau^*_{w_i}(\mathcal{L}) = \bar\nu^*_{w_i}(\mathcal{L}) \ge \vec w_i \cdot \vec z > a_i \ge \lambda \max_{\vec y\in P(\mathcal{C})} \vec w_i \cdot \vec y = \lambda\bar\nu_{w_i}(\mathcal{L})$$

for $w_i \ge 0$, a contradiction to the choice of λ.

To show the inverse inequality, for any $w \in \mathbb{R}^V_{\ge 0}$, let $x \in R(\mathcal{L})$ be a matroidal fractional matching satisfying $w \cdot x = \bar\nu^*_w(\mathcal{L})$. Then by the assumption $\frac{x}{\gamma} \in P(\mathcal{C})$ for $\mathcal{C} = \bigcap \mathcal{L}$, which means there exists a collection of $\{S_i \in \mathcal{C} : i \in I\}$ such that $\frac{x}{\gamma} = \sum_{i\in I} \lambda_i \mathbf{1}_{S_i}$. We take S_t in the collection such that $\mathbf{1}_{S_t} \cdot w = \max_{i\in I} \mathbf{1}_{S_i} \cdot w$. Therefore

$$\frac{\bar\nu^*_w(\mathcal{L})}{\gamma} = \frac{x \cdot w}{\gamma} = \sum_{i\in I} \lambda_i \mathbf{1}_{S_i} \cdot w \le \mathbf{1}_{S_t} \cdot w \le \bar\nu_w(\mathcal{L}),$$

as desired. $\square$

Next we compare $R(\mathcal{L})$ with $Q(\bigcap \mathcal{L})$.

Theorem 10.7. *For every k and $\mathcal{L} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\} \in \mathcal{D}^k$,*

$$R(\mathcal{L}) : Q(\bigcap \mathcal{L}) = \max_{U:U\subseteq V} \bar\nu^*(\mathcal{L}_U) : \bar\nu(\mathcal{L}_U),$$

where $\mathcal{L}_U = \{\mathcal{M}_1[U], \dots, \mathcal{M}_k[U]\}$.

Proof. Let $\mathcal{L} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\}$ and $\lambda = \max_{U:U\subseteq V} \bar\nu^*(\mathcal{L}_U) : \bar\nu(\mathcal{L}_U)$.

Claim 10.8. $R(\mathcal{L}) : Q(\bigcap \mathcal{L}) \le \lambda$.

Proof of the claim. Let $x \in R(\mathcal{L})$. We need to show that $x \in \lambda Q(\mathcal{C})$, where $\mathcal{C} = \bigcap \mathcal{L}$, namely

$$x[U] \leq \lambda \cdot rank_{\mathcal{C}}(U) \tag{23}$$

for every $U \subseteq V$.

Fix some $U \subseteq V$ and assume $\mathcal{L}_U = \{\mathcal{N}_1, \dots, \mathcal{N}_k\}$, where $\mathcal{N}_i = \mathcal{M}_i[U]$. We define a vector $\bar{x} \in \mathbb{R}^V_{\geq 0}$ as the following: $\bar{x}(v) = x(v)$ for $v \in U$ and $\bar{x}(v) = 0$ for $v \in V \setminus U$. Then $x \in R(\mathcal{L}) = \cap_{i=1}^k Q(\mathcal{M}_i)$ implies that for every $W \subseteq V$,

$$\bar{x}[W] = x[W \cap U] \leq \min_{1 \leq i \leq k} rank_{\mathcal{M}_i}(W \cap U) = \min_{1 \leq i \leq k} rank_{\mathcal{N}_i}(W).$$

Thus $\bar{x} \in R(\mathcal{L}_U) = \cap_{i=1}^k Q(\mathcal{N}_i)$. By the assumption of λ, $\bar{\nu}^*(\mathcal{L}_U) \leq \lambda \bar{\nu}(\mathcal{L}_U) = \lambda \cdot rank_{\mathcal{C}}(U)$, and then

$$x[U] = \bar{x}[U] \leq \bar{\nu}^*(\mathcal{L}_U) \leq \lambda \bar{\nu}(\mathcal{L}_U) = \lambda \cdot rank_{\mathcal{C}}(U),$$

as desired in (23). □

Let $\gamma = R(\mathcal{L}) : Q(\bigcap \mathcal{L})$.

Claim 10.9. $\max_{U : U \subseteq V} \bar{\nu}^*(\mathcal{L}_U) : \bar{\nu}(\mathcal{L}_U) \leq \gamma$, *i.e.,*

$$\bar{\nu}^*(\mathcal{L}_U) \leq \gamma \bar{\nu}(\mathcal{L}_U) \tag{24}$$

for every $U \subseteq V$.

Proof of the claim. Fix $U \subseteq V$ and let $x \in R(\mathcal{L}_U)$ satisfying $x[V] = \bar{\nu}^*(\mathcal{L}_U)$. By the construction of $\mathcal{L}_U$, for every $W \subseteq V$,

$$x[W] = x[W \cap U] \leq \min_{1 \leq i \leq k} rank_{\mathcal{M}_i[U]}(W \cap U) \leq \min_{1 \leq i \leq k} rank_{\mathcal{M}_i}(W),$$

which implies $x \in R(\mathcal{L}) = \cap_{i=1}^k Q(\mathcal{M}_i)$. Then by the assumption, $\frac{x}{\gamma} \in Q(\bigcap \mathcal{L})$. Therefore for every $W \subseteq V$,

$$\frac{x[W]}{\gamma} = \frac{x[W \cap U]}{\gamma} \leq rank_{\bigcap \mathcal{L}}(W \cap U) = rank_{\bigcap \mathcal{L}_U}(W),$$

which implies $\frac{x}{\gamma} \in Q(\bigcap \mathcal{L}_U)$. Therefore

$$\frac{\bar{\nu}^*(\mathcal{L}_U)}{\gamma} = \frac{x[V]}{\gamma} \leq rank_{\cap \mathcal{L}_U}(V) = \bar{\nu}(\mathcal{L}_U),$$

as desired in (24). □

This completes the proof of the theorem. □

By Theorem 10.7, Conjecture 1.8 implies the following.

Conjecture 10.10. *If* $\mathcal{L} \in \mathcal{D}^k$, *then*

$$R(\mathcal{L}) : Q(\bigcap \mathcal{L}) \leq k - 1.$$

At the risk of tiring the reader (no coercion to read) we mention two further generalizations, and their fractional cooperative covers counterparts.

Conjecture 10.11. *For any* $\mathcal{L} \in \mathcal{D}^k$

$$\bar{\tau}^*_w(\mathcal{L}) \leq (k-1)\bar{\nu}_w(\mathcal{L}).$$

Or even stronger.

Conjecture 10.12. *If $\vec{w} \in \mathbb{Z}_{\geq 0}^V$, then*

$$\bar{\tau}_w(\mathcal{L}) \leq (k-1)\bar{\nu}_w(\mathcal{L}).$$

By Theorems 10.6 and 10.4, Conjecture 10.11 is equivalent to Conjecture 9.5, and thus to Conjecture 2.3. Thus Conjecture 10.11 is stronger than Conjecture 10.10, but Conjecture 10.11 is not comparable to Conjecture 1.8. The ratio $\bar{\tau}_w^*(\mathcal{L})/\bar{\nu}_w(\mathcal{L}) = \bar{\nu}_w^*(\mathcal{L})/\bar{\nu}_w(\mathcal{L})$ is called the "integrality gap" of the linear program, so the conjecture is that the gap does not exceed $k-1$. Conjecture 10.11 is true when $k \in \{2,3\}$ by Theorem 2.1 and [26, Theorem 1]. The proof for $k=2$ case (see e.g., [31, Corolloary 41.12c]) yields the validity of Conjecture 10.12 for this k.

We conclude this section with partial results on these conjectures.

10.1. **Weaker versions of Conjecture 10.11 and Conjecture 10.12.**

Theorem 10.13. *For any $\mathcal{L} \in \mathcal{D}^k$, $\bar{\tau}_w^*(\mathcal{L}) \leq k\bar{\nu}_w(\mathcal{L})$. When $w \in \mathbb{Z}_{\geq 0}^V$, $\bar{\tau}_w(\mathcal{L}) \leq k\bar{\nu}_w(\mathcal{L})$.*

Proof. Assume $\mathcal{L} = \{\mathcal{M}_1, \cdots, \mathcal{M}_k\} \in \mathcal{D}^k$. Let $I \in \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_k$ satisfy $w[I] = \bar{\nu}_w(\mathcal{L})$ and $w(u) > 0$ for each $u \in I$. For each $1 \leq i \leq k$, we set $f_i(x) = w(x)$ if $x \in I$ and $f_i(x) = 0$ if $x \in V \setminus I$. Then $\sum_{i=1}^k f_i[V] = k\bar{\nu}_w$. It remains to verify that $\sum_{i=1}^k f_i^{\mathcal{M}_i}(v) \geq w(v)$ for each $v \in V$. Suppose not. Then there exists $v \in V$ such that $\sum_{i=1}^k f_i^{\mathcal{M}_i}(v) < w(v)$. Since for $x \in I$, $\sum_{i=1}^k f_i^{\mathcal{M}_i}(x) \geq \sum_{i=1}^k f_i(x) = kw(x)$, this v is not in I. Let J be the set of indices j such that $v \in span_{\mathcal{M}_j}(I)$. Then $\sum_{i=1}^k f_i^{\mathcal{M}_i}(v) = \sum_{j \in J} f_j^{\mathcal{M}_j}(v) < w(v)$. It means for each $j \in J$ there exists circuit C_j in $\mathcal{M}_j$ and $u_j \in C_j \setminus \{v\}$ such that $v \in C_j$ and $C_j \setminus \{v\} \subseteq I$, which satisfies

$$\sum_{j \in J} w(u_j) < w(v). \tag{25}$$

For $j \in J$, since $|C_j \setminus \{u_j\}| \leq |I|$, by the independence augmentation axiom of $\mathcal{M}_j$, $C_j \setminus \{u_j\}$ can extend to $I \cup \{v\} \setminus \{u_j\} \in \mathcal{M}_j$. For $t \notin J$, since $v \notin span_{\mathcal{M}_t}(I)$, we have $I \cup \{v\} \in \mathcal{M}_t$. Therefore

$$I' := I \cup \{v\} \setminus \{u_j \mid j \in J\} \in \cap_{i=1}^k \mathcal{M}_i = (\cap_{j \in J} \mathcal{M}_j) \cap (\cap_{t \in [k] \setminus J} \mathcal{M}_t).$$

But by (25), $w(I') \geq w(I) - \sum_{j \in J} w(u_j) + w(v) > w(I)$, contradictory to the maximum weight assumption of I, which completes the proof of the first part of the theorem.

The same proof works when $w \in \mathbb{Z}_{\geq 0}^V$, in which case $(f_1, \ldots, f_k)$ is an integral w-cooperative cover. □

By Theorem 10.6 and Corollary 8.4, the theorem is equivalent to the following.

Corollary 10.14. *For any $\mathcal{L} \in \mathcal{D}^k$, $R(\mathcal{L}) : P(\bigcap \mathcal{L}) \leq k$.*

Corollary 10.15. *For any $\mathcal{L} = \{\mathcal{M}_1, \cdots, \mathcal{M}_k\} \in \mathcal{D}^k$ and $\mathcal{C} = \bigcap \mathcal{L}$,*

$$\chi^*(\mathcal{C}, \vec{w}) \leq k \max_{1 \leq i \leq k} \chi^*(\mathcal{M}_i, \vec{w}).$$

Example 10.16. The following example shows the necessity of the requirement that $\mathcal{M}_1, \cdots, \mathcal{M}_k$ are matroids. Let $\mathcal{C}_1$ be the collection of rows and their subsets in the $n \times n$ grid, and $\mathcal{C}_2$ the collection of columns and their subsets. Then $(P(\mathcal{C}_1) \cap P(\mathcal{C}_2)) : P(\mathcal{C}_1 \cap \mathcal{C}_2) = n$.

10.2. **The intersection of partition matroids.**

Definition 10.17. Let $\mathcal{H}$ be a hypergraph and $w : \mathcal{H} \to \mathbb{R}_{\geq 0}$ be a function.

(1) A *fractional matching* in the hypergraph $\mathcal{H}$ is a function $g : \mathcal{H} \to \mathbb{R}_{\geq 0}$ satisfying
$$\sum_{e: v \in e} g(e) \leq 1$$
for every $v \in V(\mathcal{H})$. By $\nu^*_w(\mathcal{H})$ we denote the maximum value $\sum_{e \in \mathcal{H}} w(e)g(e)$ over all fractional matchings g. Furthermore $\nu_w(\mathcal{H}), \nu(\mathcal{H})$ are defined according to Remark 10.2.

(2) A *fractional w-cover* is a non-negative function t on $V(\mathcal{H})$ satisfying
$$\sum_{v \in e} t(v) \geq w(e)$$
for all $e \in \mathcal{H}$. By $\tau^*_w(\mathcal{H})$ we denote the minimum $g[V]$ over all fractional w-covers g. Furthermore $\tau_w(\mathcal{H}), \tau(\mathcal{H})$ are defined according to Remark 10.2.

The following is a weighted generalization of Ryser's conjecture.

Conjecture 10.18 (Weighted Ryser)**.** *Let $\mathcal{H}$ be a k-partite hypergraph, and let w be a non-negative integral function on $E(\mathcal{H})$. Then $\tau_w(\mathcal{H}) \leq (k-1)\nu_w(\mathcal{H})$.*

The conjecture was proved in [1] in the case $k = 3$ and $w \equiv 1$, using Theorem 3.13. A fractional version of the weighted case was proved in [15]. So far, topological methods have not been successfully applied to the weighted case.

Füredi proved a fractional version for general k.

Theorem 10.19 ([14])**.** *In a k-partite hypergraph $\nu^*(\mathcal{H}) \leq (k-1)\nu(\mathcal{H})$.*

As a treat, we give a nice proof of Theorem 10.19, by Ori Kfir in his M.Sc thesis [22].

Remark 10.20. The proof yields the result also for "bipartite" hypergraphs, namely those having a set that meets every edge at one vertex. This is not an innovation, since Füredi's original theorem contains also this case, because the condition it needs is non-containment of a projective plane, which obviously holds for "bipartite" hypergraphs.

A standard argument of adding dummy elements yields a deficiency version of Theorem 3.13.

Theorem 10.21. *Let $V_1, \ldots, V_m$ sets of vertices of a complex $\mathcal{C}$. If $\eta(\mathcal{C}[\cup_{i \in I} V_i]) \geq |I| - d$ for every $I \subseteq [m]$ then there exists a partial transversal of size $m - d$, i.e., a choice function ϕ from $S \subseteq [m]$ of size $(m - d)$ to $\bigcup_{i \in S} F_i$, such that $\phi(i) \in F_i$ for all $i \in S$ and $Im(\phi) \in \mathcal{C}$.*

Theorem 10.22 (Theorem 7.8 in [22])**.** *Let $\mathcal{H}$ be a k-uniform hypergraph. Suppose the vertex set of $\mathcal{H}$ can be divided into U and U' such that for every edge $S \in \mathcal{H}$, $|S \cap U| = 1$. Then*
$$\nu^*(\mathcal{H}) \leq (k-1)\nu(\mathcal{H}).$$

Proof. Let $\mathcal{H}' = \mathcal{H}/U = \{S \setminus U \mid S \in \mathcal{H}\}$, which is a $(k-1)$-uniform hypergraph. For every $u \in U$ let $V_u = \{S \in \mathcal{H}' \mid S \cup \{u\} \in \mathcal{H}\}$. Let
$$d = \max_{W \subseteq U} \Big(|W| - \eta(\mathcal{M}[\cup_{w \in W} V_w]) \Big),$$

and let Z be the subset of U at which the maximum is attained. Putting $W = \emptyset$ shows $d \geq 0$. By Theorem 10.21, $\nu(\mathcal{H}) \geq |U| - d$. On the other hand, by our choice, $\eta(\mathcal{M}[\cup_{w\in Z} V_w]) = |Z| - d$. By Theorem 3.10 $\eta(\mathcal{M}[\cup_{w\in Z} V_w]) \geq \tau^*(\cup_{w\in Z} V_w)$. Hence

$$(k-1)\eta(\mathcal{M}[\cup_{w\in Z} V_w]) = (k-1)(|Z| - d) \geq \tau^*(\cup_{w\in Z} V_w).$$

Let f be a fractional cover of $\cup_{w\in Z} V_w$ of total weight at most $(k-1)(|Z| - d)$. Then $f + \mathbf{1}_{U\setminus Z}$ is a fractional cover of $\mathcal{H}$, which has total weight at most

$$(k-1)(|Z|-d)+|U|-|Z| \leq (k-1)(|Z|-d+|U|-|Z|) = (k-1)(|U|-d) = (k-1)\nu(\mathcal{H}).$$

Therefore $\nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leq (k-1)\nu(\mathcal{H})$. □

Füredi, Kahn, and Seymour proved a weighted version of Theorem 10.19.

Theorem 10.23 ([15])**.** *If $\mathcal{H}$ is k-partite then for every $w : \mathcal{H} \to \mathbb{R}_{\geq 0}$, $\nu^*_w(\mathcal{H}) \leq (k-1)\nu_w(\mathcal{H})$.*

Observation 10.24. *If $\mathcal{L} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\} \in \mathcal{D}^k$ is a k-set of partition matroids and $\mathcal{H} = \mathcal{K}(\mathcal{L})$, then*

$$\bar{\tau}^*_w(\mathcal{L}) = \tau^*_w(\mathcal{H}) \quad \textit{and} \quad \bar{\nu}_w(\mathcal{L}) = \nu_w(\mathcal{H}).$$

And when $w \in \mathbb{Z}^V_{\geq 0}$, $\bar{\tau}_w(\mathcal{L}) = \tau_w(\mathcal{H})$.

Applying Theorem 10.23 and Observation 10.24, we prove Conjecture 10.11 and thus also Conjecture 9.5 and Conjecture 2.3, for partition matroids.

Corollary 10.25. *If $\mathcal{L} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\} \in \mathcal{D}^k$ is a k-set of partition matroids, then*

$$\bar{\tau}^*_w(\mathcal{L}) \leq (k-1)\bar{\nu}_w(\mathcal{L}).$$

Example 10.26 (Showing sharpness of Conjecture 10.11, Conjecture 9.5, and Conjecture 2.3)*.* The truncated projective plane T_k of uniformity k is the induced subhypergraph $P_k[V(P_k) \setminus \{v\}]$ for any vertex v of a projective plane P_k (assuming it exists). It is well-known that T_k is k-partite and each part is of size $k-1$, each vertex is contained in $k-1$ edges, $\nu(T_k) = 1$, and $\nu^*(T_k) = \tau(T_k) = k-1$. Applying Observation 10.24 and Theorem 10.4 proves the sharpness of Conjecture 10.11, and hence of the other two conjectures.

Remark 10.27*.* By Theorem 1.3, Remark 9.2 (or Example 9.3) gives a complex $\mathcal{C}$ such that $P(\mathcal{C}) \subsetneqq Q(\mathcal{C})$ and $\mathcal{C} = \bigcap \mathcal{L}_1$ for $\mathcal{L}_1 = \{\mathcal{M}_1, \dots, \mathcal{M}_{k_1}\} \in MINT_{k_1}$. Example 10.26 gives a complex $\mathcal{D}$ such that $Q(\mathcal{D}) \subsetneqq R(\mathcal{L}_2)$ and $\mathcal{D} = \bigcap \mathcal{L}_2$ for $\mathcal{L}_2 = \{\mathcal{N}_1, \dots, \mathcal{N}_{k_2}\} \in MINT_{k_2}$. Let $\mathcal{B} = \mathcal{C} * \mathcal{D}$. Then $\mathcal{B} = \bigcap \mathcal{L}$ for a $(k_1 + k_2)$-set $\mathcal{L} = \{\mathcal{M}'_1, \dots, \mathcal{M}'_{k_1}, \mathcal{N}'_1, \dots, \mathcal{N}'_{k_2}\} \in MINT_{k_1+k_2}$, where $\mathcal{M}'_i = \mathcal{M}_i * 2^{V(\mathcal{D})}$ and $\mathcal{N}'_j = 2^{V(\mathcal{C})} * \mathcal{N}_j$ for each $i \in [k_1]$ and $j \in [k_2]$. Then it is easy to see that $P(\mathcal{B}) \subsetneqq Q(\mathcal{B}) \subsetneqq R(\mathcal{L})$.

11. Fractional coloring and weighted topological expansion

The following was proved in [2].

Theorem 11.1. *For a complex $\mathcal{C}$, $\chi^*(\mathcal{C}) \leq \Delta(\mathcal{C})$. When $\mathcal{C}$ is a matroid, equality holds.*

In this section, we generalize this result to vertex-weighted complexes. For a complex $\mathcal{C}$, let

$$\Delta(\mathcal{C}, \vec{w}) := \max_{\emptyset \neq S \subseteq V(\mathcal{C})} \frac{w[S]}{\min\Big(\eta(\mathcal{C}[S]),\ rank_{\mathcal{C}}(S)\Big)}.$$

Note that $\Delta(\mathcal{C}) = \Delta(\mathcal{C}, \vec{1})$.

Remark 11.2. For a loopless matroid $\mathcal{M}$,

$$\Delta(\mathcal{M}, \vec{w}) = \max_{\emptyset \neq S \subseteq V(\mathcal{M})} \frac{w[S]}{rank_{\mathcal{M}}(S)}.$$

Theorem 11.1 is generalizable as follows. The proof involves carefully setting disjoint copies of the complexes and applying Theorem 3.13.

Theorem 11.3. $\chi^*(\mathcal{C}, \vec{w}) \leq \Delta(\mathcal{C}, \vec{w})$.

Proof. We have $\chi^*(\mathcal{C}, t\vec{w}) = t\chi^*(\mathcal{C}, \vec{w})$ and $\Delta(\mathcal{C}, t\vec{w}) = t\Delta(\mathcal{C}, \vec{w})$. Approximating $\vec{w}$ by rational vectors we may therefore assume that $\vec{w}$ is integral, thus $\Delta(\mathcal{M}, \vec{w}) = \frac{p}{q}$ for integers $p \geq 0$ and $q > 0$. Furthermore, we may assume that for every $v \in V$, $rank_{\mathcal{C}}(\{v\}) = 1$, since otherwise $\Delta(\mathcal{C}, \vec{w}) = \infty$ and the inequality holds.

Let W consist of p disjoint copies of $V(\mathcal{C})$. We consider the join complex $\mathcal{D} = *_p\mathcal{C}$ on W, and we consider a generalized partition matroid $\mathcal{N}$ on W defined in the following way: for each $v \in V(\mathcal{C})$, the set W_v consisting of the p copies of v in W is a part, and the respective constraint parameter is $w(v)q$. Since $w(v) = \frac{w(v)}{rank_{\mathcal{C}}(\{v\})} \leq \Delta(\mathcal{C}, \vec{w}) = \frac{p}{q}$, we have $w(v)q \leq p$. Thus $\mathcal{N}$ is well-defined.

If there is a base S of $\mathcal{N}$ that is in $\mathcal{D}$, then S corresponds to p faces $S_1, \ldots, S_p \in \mathcal{C}$ such that each $v \in V(\mathcal{C})$ is in $w(v)q$ of them. Then putting weight $\frac{1}{q}$ on each of the faces $S_1, \ldots, S_p$ gives an $\vec{w}$-fractional coloring of total weight $\frac{p}{q} = \Delta(\mathcal{C}, \vec{w})$, which completes the proof.

It remains to prove that there exists a base of $\mathcal{N}$ belonging to $\mathcal{D}$. Recall that the dual matroid $\mathcal{N}^*$ of $\mathcal{N}$ has the same ground set as $\mathcal{N}$, bases that are complements of bases of $\mathcal{N}$. Then $\mathcal{N}^*$ is also a generalized partition matroid with parts $(W_v)_{v \in V(\mathcal{C})}$ and respective constraint parameters $(p - w(v)q)_{v \in V(\mathcal{C})}$.

Let $U = \bigsqcup_2 W$ consist of two disjoint copies of W, and consider the complex $\mathcal{D} * \mathcal{N}^*$ on U. For each $z \in W$, let U_z be the set of two copies of z in U. If we can show that there exists a choice function $\phi : W \to \cup_{z \in W} U_z$ such that $\phi(z) \in U_z$ for each $z \in W$ and $Im(\phi) \in \mathcal{D} * \mathcal{N}^*$, then it corresponds to a set $A \subseteq W$ such that $A \in \mathcal{D}$ and $W \setminus A \in \mathcal{N}^*$. This implies that A contains a set S that is in $\mathcal{D}$ and is a base in $\mathcal{N}$. This proves the claim.

To show the existence of the choice function ϕ, we apply Theorem 3.13. The condition to be checked is that for every $X \subseteq W$,

$$\eta\Big((\mathcal{D} * \mathcal{N}^*)[\cup_{z \in X} U_z]\Big) \geq |X|.$$

By Theorem 3.4,

$$\eta\Big((\mathcal{D} * \mathcal{N}^*)[\cup_{z \in X} U_z]\Big) \geq \eta(\mathcal{D}[X]) + \eta(\mathcal{N}^*[X])$$

so it is enough to show that for every $X \subseteq W$,

$$\eta(\mathcal{D}[X]) + \eta(\mathcal{N}^*[X]) \geq |X|. \tag{26}$$

Fix some $X \subseteq W$. For each $v \in V(\mathcal{C})$, let

$$x(v) = |X \cap W_v|$$

be the number of copies of v occurring in X. Clearly $0 \leq x(v) \leq p$ and $\sum_{v \in V(\mathcal{C})} x(v) = |X|$. For each $1 \leq i \leq p$, let X_i be the intersection of X with the ith copy of $V(\mathcal{C})$. Therefore X is the disjoint union of X_i's. By the definition of $\Delta(\mathcal{C}, \vec{w})$, we have

$$\frac{p}{q} = \Delta(\mathcal{C}, \vec{w}) \geq \frac{w[X_i]}{\eta(\mathcal{C}[X_i])}$$

so that $\eta(\mathcal{C}[X_i]) \geq \frac{q}{p} w[X_i]$. Again by Theorem 3.4,

$$\eta(\mathcal{D}[X]) \geq \sum_{i=1}^{p} \eta(\mathcal{C}[X_i]) \geq \sum_{i=1}^{p} \frac{q}{p} w[X_i] = \sum_{v \in V(\mathcal{C})} \frac{q}{p} x(v) \cdot w(v). \tag{27}$$

On the other hand, by Theorem 3.5

$$\begin{aligned} \eta(\mathcal{N}^*[X]) \geq rank_{\mathcal{N}^*}(X) &= \sum_{v \in V(\mathcal{C})} \min\Big(p - w(v)q,\ x(v)\Big) \\ &\geq \sum_{v \in V(\mathcal{C})} (p - w(v)q) \frac{x(v)}{p}. \end{aligned} \tag{28}$$

Combining (27) and (28), we have

$$\eta(\mathcal{D}[X]) + \eta(\mathcal{N}^*[X]) \geq \sum_{v \in V(\mathcal{C})} x(v) = |X|,$$

which proves (26). □

Theorem 11.4. *For any complex $\mathcal{C}$,*

$$\chi^*(\mathcal{C}, \vec{w}) \geq \Delta_r(\mathcal{C}, \vec{w}) := \max_{\emptyset \neq S \subseteq V} \frac{w[S]}{rank_{\mathcal{C}}(S)}.$$

Proof. Let $X \subseteq V$ satisfy $\frac{w[X]}{rank_{\mathcal{C}}(X)} = \Delta_r(\mathcal{C}, \vec{w})$. Let f be any $\vec{w}$-fractional coloring of $\mathcal{C}$. Then

$$\sum_{v \in X} \sum_{S \in \mathcal{M}: v \in S} f(S) \geq w[X].$$

On the other hand,

$$\sum_{v \in X} \sum_{S \in \mathcal{C}: v \in S} f(S) \leq \sum_{S \in \mathcal{C}} \sum_{v \in X \cap S} f(S) \leq rank_{\mathcal{C}}(X) \sum_{S \in \mathcal{C}} f(S),$$

which implies $\sum_{S \in \mathcal{C}} f(S) \geq \frac{w[X]}{rank_{\mathcal{C}}(X)} = \Delta_r(\mathcal{C}, \vec{w})$. □

Combining Theorem 11.3, Remark 11.2, and Theorem 11.4, we have the following result.

Corollary 11.5. *For any matroid $\mathcal{M}$, $\chi^*(\mathcal{M}, \vec{w}) = \Delta(\mathcal{M}, \vec{w})$.*

12. Fractional list colorings of complexes

We conclude with a combination of the two themes of the paper, list colorings and fractional colorings. In [6] the notion of fractional list coloring of graphs was introduced. Here we generalize it to complexes. Given a complex $\mathcal{C}$, we say that $\mathcal{C}$ is (a,b)-*colorable* for some inegers $a \geq b \geq 1$, if for $L_v = \{1,\ldots,a\}$ for each $v \in V(\mathcal{C})$, there are subsets $C_v \subseteq L_v$ with $|C_v| = b$ for each $v \in V(\mathcal{C})$ such that $S_i := \{v \in V(\mathcal{C}) : i \in C_v\}$ is in $\mathcal{C}$ for each $i \in \{1,\ldots,a\}$. Let $CL(\mathcal{C}) := \{(a,b) : \mathcal{C}$ is (a,b)-colorable$\}$ and let the *colorable ratio* be $clr(\mathcal{C}) := \inf\{\frac{a}{b} : (a,b) \in CL(\mathcal{C})\}$.

Clearly, $(\chi(\mathcal{C}),1) \in CL(\mathcal{C})$ so that $clr(\mathcal{C}) \leq \chi(\mathcal{C})$.

Given a complex $\mathcal{C}$, we say that $\mathcal{C}$ is (a,b)-*choosable* for some inegers $a \geq b \geq 1$, if for any size a lists $(L_v : v \in V(\mathcal{C}))$, there are subsets $C_v \subseteq L_v$ with $|C_v| = b$ for each $v \in V(\mathcal{C})$ such that $S_i := \{v \in V(\mathcal{C}) : i \in C_v\}$ is in $\mathcal{C}$ for each $i \in \cup_{v\in V(\mathcal{C})} L_v$. Let $CH(\mathcal{C}) := \{(a,b) : \mathcal{C}$ is (a,b)-choosable$\}$ and let the *choice ratio* be $chr(\mathcal{C}) := \inf\{\frac{a}{b} : (a,b) \in CH(\mathcal{C})\}$.

Clearly, $(\chi_\ell(\mathcal{C}),1) \in CH(\mathcal{C})$, so $chr(\mathcal{C}) \leq \chi_\ell(\mathcal{C})$.

Theorem 12.1. *For any complex $\mathcal{C}$, $\chi^*(\mathcal{C}) = clr(\mathcal{C}) = chr(\mathcal{C})$.*

The proof is similar to that in [6], and we sketch here for completeness.

Lemma 12.2. *For any complex $\mathcal{C}$, $clr(\mathcal{C}) \geq \chi^*(\mathcal{C})$.*

The proof is essentially same as the proof of lower bound in [6, Section 4].

Lemma 12.3. *For a complex $\mathcal{C}$, $chr(\mathcal{C}) \geq clr(\mathcal{C})$.*

We omit the straightforward proof.

Lemma 12.4. *For a complex $\mathcal{C}$, $\chi^*(\mathcal{C}) \geq chr(\mathcal{C})$.*

Proof. The first observation is that the optimum of the linear program of fractional coloring can be obtained by some basic feasible solution f, and since the constraints of fractional coloring are integral, by Cramer's rule, there exists a subset $\mathcal{C}_0$ of $\mathcal{C}$ and a positive integer q such that for each $T \in \mathcal{C}_0$, $f(T) = p_T/q$ for some positive integer p_T, and for each $T \in \mathcal{C} \setminus \mathcal{C}_0$, $f(T) = 0$. And

$$\chi^*(\mathcal{C}) = f[\mathcal{C}] = f[\mathcal{C}_0] = \frac{p}{q},$$

where $p := \sum_{T\in\mathcal{C}_0} p_T$. By taking each $T \in \mathcal{C}_0$ p_T times, we obtain a collection of faces $\{T_1,\ldots,T_p\}$ (not necessarily distinct) such that each vertex $v \in V(\mathcal{C})$ is in at least q many of them:

$$|\{j : v \in T_j\}| = \sum_{T\in\mathcal{C}_0 : v\in T} p_T = q \sum_{T\in\mathcal{C} : v\in T} f(T) \geq q,$$

where the last inequality is by the constraint of the linear program.

Then to prove $\frac{p}{q} = \chi^*(\mathcal{C}) \geq chr(\mathcal{C})$, we have to show that for every $\epsilon > 0$ there exists $(a,b) \in CH(\mathcal{C})$ such that $\frac{a}{b} \leq (1+\epsilon)\frac{p}{q}$. Consider $a := (1+\epsilon)pt$ and $b := qt$, for t sufficiently large, and we can assume without loss of generality that a, b are integers. For any size a lists $(L_v : v \in V(\mathcal{C}))$, we take a random partition $\cup_{v\in V(\mathcal{C})} L_v = Z_1 \cup \ldots Z_p$ such that each color is included in each Z_j independently with probability $1/p$. Noting that p and $|V(\mathcal{C})|$ are fixed for any given $\mathcal{C}$, it is followed by the well-known Chernoff's bounds that with positive probability, for each $v \in V(\mathcal{C})$ and $j \in \{1, \ldots, p\}$, $|L_v \cap Z_j|$ is concentrated around its expectation $|L_v|/p = (1+\epsilon)t$. Thus we can take an t-subset $C_{v,j}$ of $L_v \cap Z_j$. Then we get a set

$$C_v := \cup_{1 \le j \le p: v \in T_j} C_{v,j}$$

of size at least $qt = b$.

Then for each color $i \in \cup_{v \in V(\mathcal{C})} L_v$ and $S_i = \{v \in V(\mathcal{C}) \mid i \in C_v\}$, if $i \notin \cup_{v \in V(\mathcal{C})} C_v$, then $S_i = \emptyset$, otherwise $S_i \subseteq T_j$ for the index j satisfying $i \in Z_j$. Since $T_j \in \mathcal{C}$, in either case $S_i \in \mathcal{C}$ and we prove $(a, b) \in CH(\mathcal{C})$. □

Combining the above lemmas, we have

$$\chi^*(\mathcal{C}) = chr(\mathcal{C}) = clr(\mathcal{C}),$$

which completes the proof of Theorem 12.1.

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