SHORT RAINBOW CYCLES FOR FAMILIES OF SMALL EDGE SETS

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Abstract. In 2019, Aharoni proposed a conjecture generalizing the Caccetta-Häggkvist conjecture: if an n-vertex graph G admits an edge coloring (not necessarily proper) with n colors such that each color class has size at least r, then G contains a rainbow cycle of length at most $\lceil n/r \rceil$. Recent works [3, 1, 8] have shown that if a constant fraction of the color classes are non-star, then the rainbow girth is $O(\log n)$. In this note, we extend these results, and show that even a small fraction of non-star color classes suffices to ensure logarithmic rainbow girth. We also prove that the logarithmic bound is of the right order of magnitude. Moreover, we determine the threshold fraction between the types of color classes at which the rainbow girth transitions from linear to logarithmic.

1. Introduction

The following is a well-known conjecture in graph theory by Caccetta and Häggkvist [5] from 1978.

Conjecture 1 ([5])**.** *In an n-vertex digraph with minimum out-degree r, there exists a directed cycle of length at most $\lceil \frac{n}{r} \rceil$.*

The conjecture is still open. For a history of the studying of this problem, see, e.g., the references in [11, 12, 13].

Given a graph G and an edge coloring (not necessarily proper) with m colors $\lambda : E(G) \to [m]$, for each $i \in [m]$ the edge set $\lambda^{-1}(i)$ is called a *color class*, and a subgraph H of G is *rainbow* if no two edges of H belong to the same color class. The *rainbow girth* of G is the minimum length of a rainbow cycle in G (∞ if there is no rainbow cycle). In [2], a generalization of the Caccetta-Häggkvist conjecture is raised by Aharoni.

Conjecture 2 ([2])**.** *For an n-vertex graph G and an edge coloring of G with n colors, if each color class is of size at least r, then the rainbow girth at most $\lceil \frac{n}{r} \rceil$.*

A more general version of the conjecture (see, e.g., [1, Conjecture 1.5]) states that for a family $(F_1, \dots, F_n)$ of edge sets on a vertex set of size n, where each F_i has size r and the sets are not necessarily disjoint, there exists a cycle of length at most $\lceil \frac{n}{r} \rceil$ in which each edge comes from a distinct set in the family. For our purposes, we may assume that the sets F_i are disjoint: otherwise, there exists a digon, i.e., a cycle of length 2, whose edges are from distinct sets.

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See [3] for a proof that Conjecture 2 implies Conjecture 1. And if Conjecture 2 is true, then the bound is tight: Consider the n-vertex graph on $\{x_1, \dots, x_n\}$ and each color class forms a star of size r with the form $\{x_i x_{i+1}, \dots, x_i x_{i+r}\}$ for $1 \le i \le n$, where the indices are taken modulo n.

Regarding Conjecture 2, DeVos et al. [7] proved that the conjecture is true when $r = 2$. For the case that each color class is of size 1 or 2, a more explicit bound on the rainbow girth is given in [1].

Theorem 1 ([1])**.** *For an n-vertex graph and an edge coloring with n colors, if p color classes are of size 1 and $n - p$ color classes are of size 2, then the rainbow girth is at most $\lceil \frac{n+p}{2} \rceil$.*

The $r = 3$ case of Conjecture 2 is still open, while Clinch et al. [6] proved that for $r = 3$ the rainbow girth is at most $\frac{4n}{9} + 7$. For general r, Hompe and Spirkl [10] proved that there exists a constant $0 < C \le 10^{11}$ such that the rainbow girth is at most $C\frac{n}{r}$. Recently, Hompe and Huynh [9] proved a stronger result that the rainbow girth is at most $\frac{n}{r} + \alpha_r$ for some positive constant α_r depending on r.

As the bound on the rainbow girth in Conjecture 2 is linear in n and in all the known extremal examples the color classes are stars, another direction is to study when the rainbow girth is small, say, logarithmic in n.

Theorem 2 ([3])**.** *There exists $C > 0$ such that for any n-vertex graph and edge coloring with n colors, if each color class contains a matching of size 2, then the rainbow girth is at most $C \log n$.*

In [8], the result is strengthened to the following.

Theorem 3 ([8])**.** *For any $\alpha > \frac{1}{2}$, there exists $C > 0$ such that the following holds. Let G be an n-vertex graph admitting an edge coloring with n colors such that each color class is non-empty. If there are at least αn color classes, each of which contains a matching of size 2, then the rainbow girth of G is at most $C \log n$.*

The example in [8, Section 3.1.2], which is an n-vertex graph satisfying all the other conditions in Theorem 3 for $\alpha = \frac{1}{2}$ and having linear rainbow girth, shows that the condition $\alpha > \frac{1}{2}$ is tight to ensure logarithmic rainbow girth.

In this note, we further strengthen the above result.

Theorem 4. *For any $\alpha, \beta \ge 0$ with $2\alpha + \beta > 1$, there exists $C > 0$ such that the following holds. Let G be an n-vertex graph admitting an edge coloring with n colors. If the family of color classes $\mathcal{F} = (F_1, \dots, F_n)$ satisfies the following conditions:*

- *each color class is non-empty,*
- *there exists $\mathcal{F}_M \subseteq \mathcal{F}$ such that $|\mathcal{F}_M| = \alpha n$ and each color class in $\mathcal{F}_M$ contains a matching of size 2,*
- *and there exists $\mathcal{F}_S \subseteq \mathcal{F} \setminus \mathcal{F}_M$ such that $|\mathcal{F}_S| = \beta n$ and each color class in $\mathcal{F}_S$ contains a star of size 2.*

Then the rainbow girth of G is at most $C \log n$.

Remark 5. *To have logarithmic rainbow girth, the condition $2\alpha + \beta > 1$ in Theorem 4 is tight. See Example 17.*

In Theorem 14, we prove a slightly stronger result, which allows the numbers of color classes to be less than those in Theorem 4. As an immediate implication, if each of the n color classes is of size at least 2, then a small fraction of color classes that contain a matching of size 2 guarantee the rainbow girth to be logarithmic.

Corollary 6. *For any $\alpha > 0$, there exists $C > 0$ such that the following holds. Let G be an n-vertex graph admitting an edge coloring with n colors such that each color class is of size at least 2. If there are at least αn color classes, each of which contains a matching of size 2, then the rainbow girth of G is at most $C \log n$.*

Since a non-star edge set that does not contain a matching of size 2 is a triangle, it was shown in [1] that for an n-vertex graph admitting an edge coloring with n colors, if each color class contains a triangle, then the rainbow girth is $O(\log n)$. This result is extended in [8] by considering a mixed case.

Theorem 7 ([8])**.** *There exits $C > 0$ such that for any n-vertex graph and edge coloring with n colors, if each color class contains either a matching of size 2 or a triangle, then the rainbow girth is at most $C \log n$.*

We further extend Corollary 6 and Theorem 7 and show that a small fraction of non-star color classes make the rainbow grith logarithmic.

Theorem 8. *For any $\alpha > 0$, there exists $C > 0$ such that the following holds. Let G be an n-vertex graph admitting an edge coloring with n colors such that each color class is of size at least 2. If there are at least αn color classes, each of which contains either a matching of size 2 or a triangle, then the rainbow girth of G is at most $C \log n$.*

Remark 9. *To have logarithmic rainbow girth, the condition $\alpha > 0$ is tight. See Remark 5 for the case $\alpha = 0$ and $\beta = 1$.*

Theorem 16 later shows that among n color classes, each of size at least 2, the presence of $\omega(\sqrt{n})$ non-star classes already forces the rainbow girth to be $o(n)$.

Noting that any set of r edges is a subset of the edge set of the complete graph K_{2r}, a special case of the following Theorem 10 (taking the uniformity $k = 2$ and setting $\mathcal{F} = (C_\ell)_{\ell \geq 2}$ as the collection of cycles) shows that the logarithmic bounds in all of the above results are of the right order of magnitude.

To state Theorem 10 concerning hypergraphs, a finite collection F of k-sets is called a *k-uniform hypergraph*, or a *k-graph* for brevity, whose *vertex set* is $V(F) := \cup_{e \in F} e$. An element of F is called a *(hyper)edge* of F. A *complete k-graph on t vertices*, denoted by $K_t^{(k)}$, is the collection of all k-subsets of the t-vertex set. The *density* of a hypergraph F is $\frac{|F|}{|V(F)|}$. A $\frac{1}{k-1}$*-dense sequence* of k-graphs is a sequence $(F_\ell)_{\ell \geq 2}$ of k-graphs such that for each ℓ, $|F_\ell| = \ell$ and the density of F_ℓ is at least $\frac{1}{k-1}$, which is equivalent to $|V(F_\ell)| \leq \ell(k-1)$. For example, $(B_\ell^{(k)})_{\ell \geq 2}$, where $B_\ell^{(k)}$ is a Berge k-cycle of length ℓ, is a $\frac{1}{k-1}$-dense sequence. For a hypergraph, the definition of color classes and rainbow subsets is just same as the graph case.

Theorem 10. *For any $L \geq 0$, $t \geq k \geq 2$, and $\delta > 0$, there exist constants $c, n_0 > 0$ such that the following holds. For any integer $n \geq n_0$ and any union $\mathcal{F}$ of at most $n^{1-\delta}$ many $\frac{1}{k-1}$-dense sequences of k-graphs, there exists an n-vertex k-graph H and an edge coloring with at least Ln colors satisfying the following conditions:*

- *each color class is $K_t^{(k)}$,*
- *and the minimum size of $F \in \mathcal{F}$ such that there exists a rainbow copy of F in H is at least $c \log n$.*

2. Proofs and examples

2.1. **Preliminaries.** For a graph H, the *excess* of H is defined as the difference between its number of edges and its number of vertices, i.e., $|E(H)| - |V(H)|$.

Bollobás and Szemerédi [4] proved the following upper bound on the girth of a graph in terms of its excess.

Theorem 11. *For all $n \geq 4$ and $k \geq 2$, every n-vertex graph with excess at least k has girth at most*

$$\frac{2(n+k)}{3k}(\log_2 k + \log_2 \log_2 k + 4). \tag{1}$$

Thus if an n-vertex graph has excess $\Omega(n)$, then its girth is $O(\log n)$.

Two probabilistic tools we will use are Chernoff's bound and Markov's inequality.

Theorem 12 (Chernoff)**.** *Let X be a binomial random variable $Bin(n,p)$. For any $\epsilon \in (0,1)$,*

$$\mathbb{P}(X \leq (1-\epsilon)\mathbb{E}X) \leq e^{-\epsilon^2 \mathbb{E}X/3}.$$

Theorem 13 (Markov)**.** *For any non-negative random variable X and $t > 0$,*

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t}.$$

2.2. **Proof of Theorem 4.** As mentioned, we prove a slightly stronger result compared to Theorem 4.

Theorem 14. *For any $\alpha, \beta \geq 0$ with $2\alpha + \beta > 1$, there exist $\xi(\alpha,\beta) > 0$ and $C(\alpha,\beta) > 0$ such that the following holds. Let G be an n-vertex graph with an edge coloring. If the family of color classes $\mathcal{F} = (F_1, \ldots, F_m)$ satisfies the following conditions:*

- *each color class is non-empty,*
- *there exists $\mathcal{F}_M \subseteq \mathcal{F}$ such that $|\mathcal{F}_M| \geq (\alpha - \xi)n$ and each color class in $\mathcal{F}_M$ contains a matching of size 2,*
- *there exists $\mathcal{F}_S \subseteq \mathcal{F} \setminus \mathcal{F}_M$ such that $|\mathcal{F}_S| \geq (\beta - \xi)n$ and each color class in $\mathcal{F}_S$ contains a star of size 2,*
- *and $|\mathcal{F} \setminus (\mathcal{F}_M \cup \mathcal{F}_S)| \geq (1 - \alpha - \beta - \xi)n$.*

Then the rainbow girth of G is at most $C \log n$.

Proof. Without loss of generality, we may assume that $\max(\alpha, \beta) \leq 1$, otherwise for $\xi = \frac{\max(\alpha,\beta)-1}{2}$, by Theorem 11 taking one arbitrary edge from each color class already yields logarithmic rainbow girth.

For some $p := 1 - t \in (0,1)$ to be determined later, let S be a p-random subset of $V(G) = [n]$, i.e., each vertex of G is included in S independently with probability p. We construct a subgraph H of G in the following way: the vertex set of H is S, and for each color class of G, we include in H one arbitrary edge from that class, provided the edge is entirely contained within S. Note that H constructed in this way is rainbow. Let

$$K = |E(H)| - |V(H)|$$

be the excess of H. We aim to show that

$$\mathbb{E}K \geq \delta n \tag{2}$$

for some $\delta(\alpha, \beta) > 0$. This implies that there exists an instance of H whose excess is at least δn, and then by the argument below Theorem 11, there exists a cycle in H, which is rainbow in G, of length at most $C \log n$ for some $C(\delta) > 0$. This will complete the proof.

To prove (2), for a matching of size 2 in G, by inclusion-exclusion, the probability that one of its edges is contained in S is $2p^2 - p^4$. For a star of size 2, the probability that one of its edges is contained in S is $2p^2 - p^3 = p(1-(1-p)^2)$. And for a single edge, the probability that it is contained in S is p^2.

Combining these probabilities with the conditions that $|\mathcal{F}_M| \geq (\alpha - \xi)n$, $|\mathcal{F}_S| \geq (\beta - \xi)n$, and $|\mathcal{F} \setminus (\mathcal{F}_M \cup \mathcal{F}_S)| \geq (1 - \alpha - \beta - \xi)n$, by linearity of expectation we have

$$\begin{aligned}\mathbb{E}K =& \mathbb{E}|E(H)| - \mathbb{E}|V(H)| \\ \geq& (\alpha - \xi)n(2p^2 - p^4) + (\beta - \xi)n(2p^2 - p^3) + (1 - \alpha - \beta - \xi)np^2 - np.\end{aligned}$$

Substituting $p = 1 - t$, it implies that

$$\begin{aligned}\mathbb{E}K \geq& (2\alpha + \beta - 1)tn + (-\alpha t^2 + 4\alpha t - 5\alpha + \beta t - 2\beta + 1)t^2 n \\ &+ (-3 + 3t + 4t^2 - 5t^3 + t^4)\xi n.\end{aligned}$$

Setting $\gamma := 2\alpha + \beta - 1 > 0$, $t := \frac{\gamma}{40} \leq 1$, and $\xi := \frac{\gamma t}{100}$, it yields

$$\mathbb{E}K \geq \gamma tn - 20 \cdot t(tn) - 20\xi n = \gamma tn - \frac{1}{2}\gamma tn - \frac{1}{5}\gamma tn \geq \frac{1}{10}\gamma tn.$$

Setting $\delta := \frac{1}{10}\gamma t$ completes the proof of (2). □

Remark 15. *The condition $2\alpha+\beta > 1$ in Theorem 14 (or Theorem 4) is equivalent to $\alpha > 1 - \alpha - \beta$, i.e., the proportion of classes that contain a matching of size 2 is greater than the proportion consisting of a single edge.*

2.3. **Proof of Theorem 8.**

Proof of Theorem 8. If there are at least $\frac{\alpha}{2}n$ color classes of G, each of which contains a matching of size 2, then Corollary 6 implies that the rainbow girth is at most $C_1 \log n$ for some constant $C_1 > 0$ depending on α.

Otherwise, there are at least $\frac{\alpha}{2}n$ color classes of G, each of which contains a triangle. We take arbitrarily two edges from a triangle in each of these color classes, and arbitrarily one edge from each of the remaining color classes. Then we get a graph F which has at least

$$\frac{\alpha}{2}n \cdot 2 + (n - \frac{\alpha}{2}n) = (1 + \frac{\alpha}{2})n$$

edges on n vertices. As the excess of F is at least $\frac{\alpha}{2}n$, Theorem 11 implies that there exists a constant $C_2(\alpha) > 0$ such that the girth of F is at most $C_2 \log n$. If the shortest cycle in F is not rainbow, two edges of the same color must come from a monochromatic triangle by the construction of F, then we can replace these two edges by the third edge of the triangle and obtain a shorter cycle in G. Do this replacement repeatedly until we get a rainbow cycle, which is a rainbow cycle in G of length at most $C_2 \log n$.

Taking $C = \max(C_1, C_2)$ completes the proof. □

The following result provides a more explicit upper bound on the rainbow girth in terms of the number of non-star color classes.

Theorem 16. *There exists $L_0 > 0$ such that for any $0 \le c \le \frac{1}{2}$ and $L \ge L_0$, the following holds for some $n_0 > 0$. For any integer $n \ge n_0$, let G be an n-vertex graph admitting an edge coloring with n colors such that each color class is of size at least 2. If there are at least Ln^{1-c} color classes, each of which contains either a matching of size 2 or a triangle, then the rainbow girth of G is at most $\frac{2\log_2(\frac{L}{10^2}n^{1-2c})}{\frac{L}{10^2}n^{1-2c}}n$.*

Proof. First note that there exists a constant $A > 0$ such that for all large n and $A \le k \le n$, the bound in (1) satisfies that

$$\frac{2(n+k)}{3k}(\log_2 k + \log_2\log_2 k + 4) \le \frac{2\log_2 k}{k}n \tag{3}$$

and the function $f(k) := \frac{\log_2 k}{k}$ is decreasing for $k \in [A, +\infty)$.

With some forecast, we set

$$L_0 := \max(100A,\ 10^3). \tag{4}$$

If at least $\frac{L}{2}n^{1-c}$ color classes of G contains a triangle, following the same way as in the proof of Theorem 8 by taking arbitrary two edges from a triangle in each of these color classes and taking one edge from each of the remaining color classes, we obtain a subgraph F with excess at least $\frac{L}{2}n^{1-c}$, which by Theorem 11 and (3) yields a rainbow cycle in G of length at most

$$\frac{2\log_2(\frac{L}{2}n^{1-c})}{\frac{L}{2}n^{1-c}}n \le \frac{2\log_2(\frac{L}{10^2}n^{1-2c})}{\frac{L}{10^2}n^{1-2c}}n,$$

where the last inequality is by the monotonicity of $f(k)$.

Otherwise, there are T color classes of G, each of which contains a matching of size 2, for some $T \ge \frac{L}{2}n^{1-c}$. Then each of the other $n - T$ color classes contains a star of size 2. Similarly as the proof of Theorem 14, for some $p := 1 - t \in (0, 1)$ to be determined later, we take a p-random subset S of $V(G)$. And for each color class, we arbitrarily take one edge contained entirely in S, if such an edge exists, to form a rainbow subgraph H with vertex set S. For the excess K of H, we have

$$\begin{aligned}\mathbb{E}K &= \mathbb{E}|E(H)| - \mathbb{E}|V(H)| \\ &\ge T(2p^2 - p^4) + (n - T)(2p^2 - p^3) - np \\ &= \Big(Tt - 3Tt^2 + 3Tt^3 - Tt^4\Big) + \Big(-nt^2 + nt^3\Big).\end{aligned}$$

Setting $t := \frac{1}{10}n^{-c} \le \frac{1}{10}$ and using $T \ge \frac{L}{2}n^{1-c}$, it implies that

$$\mathbb{E}K \ge \frac{1}{2}tT - nt^2 \ge \frac{1}{2}\cdot\frac{1}{10}n^{-c}\cdot\frac{L}{2}n^{1-c} - \frac{1}{10^2}n^{1-2c} \ge \frac{L}{10^2}n^{1-2c}.$$

Therefore there exists an instance H whose excess is at least $\frac{L}{10^2}n^{1-2c}$. Then by (4), Theorem 11, and (3), it implies that there is a cycle in H of length at most $\frac{2\log_2(\frac{L}{10^2}n^{1-2c})}{\frac{L}{10^2}n^{1-2c}}n$, which is rainbow in G. This completes the proof. □

2.4. **Proof of Theorem 10.** For a t-graph G on $[n]$, the *k-shadow* of G is

$$\partial_k(G) := \{S \in \binom{[n]}{k} \mid S \subseteq e \text{ for some } e \in G\}.$$

Proof of Theorem 10. For
$$p := \frac{4 \cdot 2^t t! L}{n^{t-1}},$$
let G_0 be $G^{(t)}(n,p)$, i.e., each edge of the complete t-graph $K_n^{(t)}$ is included in G_0 independently with probability p.

The idea to prove the theorem is that by alteration, we shall find some $G \subseteq G_0$ such that $|G| \geq Ln$, $|e \cap f| \leq 1$ for any distinct $e, f \in G$, and for $H = \partial_k(G)$ with color classes $(\binom{e}{k})_{e \in G}$, the minimum size of $F \in \mathcal{F}$ such that there is a rainbow copy of F in H is at least $c \log n$. Note that by construction, $H = \cup_{e \in G} \binom{e}{k}$, and the condition $|e \cap f| \leq 1 < k$ for distinct $e, f \in G$ guarantees that the color classes are disjoint.

Turn to the details. We assume that n is large enough in the following. Since
$$\mathbb{E}|G_0| = \binom{n}{t} p \geq \frac{(n-t)^t}{t!} p \geq \frac{(n/2)^t}{t!} p \geq 4Ln,$$
by Chernoff's bound Theorem 12, the event
$$\mathcal{A} := \{|G_0| \geq 3Ln\}$$
holds with probability at least $\mathbb{P}(|G_0| \geq 0.9\mathbb{E}|G_0|) = 1 - o(1)$.

Let Y be the number of pairs (e, f) such that $e, f \in G_0$ are distinct and $|e \cap f| \geq 2$. Since there are at most $\binom{n}{t}$ ways to choose a t-set e, at most $\binom{t}{2}$ ways to determine two vertices in the intersection, and at most $\binom{n-2}{t-2}$ ways to extend the two vertices to a t-vertex set f, we have
$$\mathbb{E}Y \leq \binom{n}{t}\binom{t}{2}\binom{n-2}{t-2} p^2 = o(n).$$
Thus Markov's inequality Theorem 13 implies that the event
$$\mathcal{B} := \{Y \leq Ln\}$$
holds with probability $1 - o(1)$.

For a copy $C = \{S_1, \ldots, S_\ell\}$ of a k-graph F of size ℓ on $[n]$ and a t-graph T on $[n]$, C is called *distinguishable in* T if there exists $e_1, \ldots, e_\ell \in T$ such that $S_i \subseteq e_i$ for each $1 \leq i \leq \ell$, and $S_j \not\subseteq e_i$ for $j \neq i$. To bound the probability of C being distinguishable in G_0, for each $1 \leq i \leq \ell$ let $\mathcal{T}_{S_i}$ be the event that there exists $e_i \in G_0$ such that $S_i \subseteq e_i$ and $S_j \not\subseteq e_i$ for all $j < i$. Then
$$\mathbb{P}(C \text{ is distinguishable in } G_0) \leq \mathbb{P}(\cap_{i=1}^{\ell} \mathcal{T}_{S_i}) = \prod_{i=1}^{\ell} \mathbb{P}(\mathcal{T}_{S_i} \mid \cap_{j=1}^{i-1} \mathcal{T}_{S_j}).$$
Since there are at most $\binom{n-k}{t-k}$ many t-sets e satisfying $S_i \subseteq e$ and $S_j \not\subseteq e$ for all $j < i$,
$$\mathbb{P}(\neg \mathcal{T}_{S_i} \mid \cap_{j=1}^{i-1} \mathcal{T}_{S_j}) \geq (1-p)^{\binom{n-k}{t-k}} \geq 1 - p\binom{n-k}{t-k},$$
where the last inequality is by Bernoulli's inequality $(1-p)^N \geq 1 - pN$ for $p \in (0,1)$ and positive integer N. Therefore
$$\mathbb{P}(C \text{ is distinguishable in } G_0) \leq \left(p\binom{n-k}{t-k}\right)^{\ell} \leq p^{\ell} n^{\ell(t-k)}.$$
Let X_ℓ be the number of distinguishable copies on $[n]$ of some k-graph in $\mathcal{F}$ of size ℓ. Given $F \in \mathcal{F}$ of size ℓ, by the density assumption of F, we have $|V(F)| \leq \ell(k-1)$, so

the number of copies of F on $[n]$ is at most $n^{\ell(k-1)}$. Because there are at most $n^{1-\delta}$ many k-graphs of size ℓ in $\mathcal{F}$,

$$\mathbb{E}X_\ell \le n^{1-\delta} \cdot n^{\ell(k-1)} p^\ell n^{\ell(t-k)} = n^{1-\delta}(4 \cdot 2^t t! L)^\ell.$$

By choosing $c(L,t,\delta) > 0$ small enough (say $c \log(4 \cdot 2^t t! L) \le \delta/9$), for $2 \le \ell \le c \log n$, we have

$$\mathbb{E}X_\ell = o(n^{1-\frac{2\delta}{3}}).$$

Thus by Markov's inequality,

$$\mathbb{P}(X_\ell \ge n^{1-\frac{\delta}{3}}) = o(n^{-\frac{\delta}{3}}).$$

Taking a union bound over all $2 \le \ell \le c \log n$, the event

$$\mathcal{C} := \{ \sum_{2 \le \ell \le c \log n} X_\ell \le Ln \}$$

holds with probability $1 - o(1)$. Summing up, we have

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) = 1 - o(1).$$

Take an instance G_0 such that the event $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$ holds. The event $\mathcal{A}$ ensures that $|G_0| \ge 3Ln$. If there two edges of G_0 intersecting at more than one vertex, we remove one of the edges. Then event $\mathcal{B}$ guarantees that we remove at most Ln edges. Let the remaining t-graph be G_1. If for some $2 \le \ell \le c \log n$, there is a copy $\{S_1, \dots, S_\ell\}$ on $[n]$ of some $F \in \mathcal{F}$ of size ℓ and $e_1, \dots, e_\ell \in G_1$ such that $S_i \subseteq e_i$ for each $1 \le i \le \ell$ and $S_j \not\subseteq e_i$ for $j \ne i$, we remove arbitrary one edge from $e_1, \dots, e_\ell$. Then event $\mathcal{C}$ guarantees that in this step we remove at most Ln edges from G_1. Let the resulting t-graph be G and let $H := \partial_k(G)$. Since $G \subseteq G_1$, $|e \cap f| \le 1$ for any distinct $e, f \in G$. And there is no distinguishable copy of F in G for $F \in \mathcal{F}$ with $|F| \le c \log n$.

Then H satisfies the conclusion of the theorem: let the color classes of H be $(\binom{e}{k})_{e \in G}$, so there are $|G| \ge |G_0| - Ln - Ln \ge Ln$ many color classes and each class is a copy of $K_t^{(k)}$. As discussed, the classes are disjoint and their union is H. Furthermore, if there is a rainbow copy of F in H for some $F \in \mathcal{F}$, then the copy is distinguishable in G, which implies that $|F| > c \log n$. □

2.5. **Tightness of the condition $2\alpha + \beta > 1$ in Theorem 4.** To get logarithmic rainbow girth in Theorem 4, it is necessary to assume that $2\alpha + \beta > 1$. For $\alpha, \beta \ge 0$ with $2\alpha + \beta = 1$, the following is an example of an n-vertex graph satisfying all the other conditions in Theorem 4, whose rainbow girth is linear in n.

Example 17. *When $\alpha = 0$ or $\beta = 0$, the tightness has been shown by the example below Conjecture 2, or the example mentioned below Theorem 3, respectively.*

For $\min(\alpha, \beta) > 0$ with $2\alpha + \beta = 1$, we assume that the integer αn is at least four and is divisible by two. We shall construct an n-vertex G and an edge coloring with n colors, such that αn color classes are matchings of size 2, βn color classes are stars of size 2, and $(1 - \alpha - \beta)n = \alpha n$ color classes are of size 1. Furthermore the rainbow girth of G is at least $\min(\alpha n, \frac{\beta}{2} n)$, which is linear in n.

The graph has two connected components on $X = \{x_1, \dots, x_{2\alpha n}\}$ and $Y = \{y_1, \dots, y_{\beta n}\}$, respectively. The αn many edge sets $\{x_{4i+1}x_{4i+2}, x_{4i+3}x_{4i+4}\}$ and $\{x_{4i+2}x_{4i+5}, x_{4i+4}x_{4i+7}\}$ for $i = 1, \dots, \frac{\alpha n}{2}$ are color classes that are matchings of size 2, where the indices are taken modulo $2\alpha n$. The αn many edge sets $\{x_{4i+1}x_{4i+3}\}$ and $\{x_{4i+2}x_{4i+4}\}$ for $i = 1, \dots, \frac{\alpha n}{2}$ are color classes consisting of a single edge, *where the indices are taken modulo* $2\alpha n$. *The* βn *many edge sets* $\{y_i y_{i+1}, y_i y_{i+2}\}$ *are color classes that are stars of size* 2, *where the indices are taken modulo* βn. *Then a rainbow cycle of* G *is either in* X *or in* Y, *which has length at least* $\min(\alpha n, \frac{\beta}{2} n)$.

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